Finite Difference Approximation of Free Discontinuity Problems

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Abstract

We approximate functionals depending on the gradient of u and on the behaviour of u near the discontinuity points, by families of non-local functionals where the gradient is replaced by finite differences. We prove pointwise convergence, Γ-convergence, and a compactness result which implies, in particular, the convergence of minima and minimizers.

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1 Introduction

In mathematical literature many free discontinuity problems have been considered. The canonical examples are the minimum problems related to the so called Mumford-Shah functional, defined by

$$MS(u) = \int_\Omega |\nabla u(x)|^2 \, dx + \mathcal{H}^{n-1}(S_u), \tag{1.1}$$

where Ω is an open subset of $\mathbb{R}^n$, u belongs to the space $SBV(\Omega)$ of special functions with bounded variation (see § 2.1), ∇u is the approximate gradient of u, S_u is the set of essential discontinuity points of u, and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure.

This functional is the weak formulation in the space $SBV(\Omega)$ of the functional introduced by D. Mumford and J. Shah in [19] to approach image segmentation problems.

A natural generalization of (1.1) are the functionals

$$\mathcal{F}(u) = \int_\Omega \varphi(|\nabla u(x)|) \, dx + \int_{S_u} \psi(|u^+(x) - u^-(x)|) \, d\mathcal{H}^{n-1}(x), \tag{1.2}$$

where $\varphi, \psi : [0, +\infty[\to [0, +\infty]$ are given functions, and $u^+(x)$ and $u^-(x)$ are the approximate (in the measure theoretic sense) $\limsup$ and $\liminf$ of u at the point x.

By the semicontinuity and compactness theorem in SBV proved by L. Ambrosio in [3], variational problems involving $\mathcal{F}$ can be solved using the direct methods of the calculus of variations: the interested reader can find appropriate references in the survey [5].

Approximations of (1.1) and (1.2) have been deeply studied in last years, both because of numerical applications, and in order to approach evolution problems with free discontinuities (cf. [18]). In this context, approximation is always required in the sense of Γ-convergence (see § 2.2), since this notion is stable under continuous perturbations, and guarantees the convergence of minima and minimizers.

It is well known (cf. [9]) that functionals like (1.1) and (1.2) *cannot* be approximated in the sense of Γ-convergence by local integral functionals like

$$\int_\Omega f_\varepsilon(\nabla u(x)) \, dx, \tag{1.3}$$

defined in the Sobolev space $W^{1,2}(\Omega)$. This difficulty has been overcome in different ways (cf. the survey [8]):

- by introducing an auxiliary variable as in [6, 7];
- by considering non-local functionals depending on the average of the gradient in small balls as in [9];
- by adding to (1.3) a singular perturbation depending on higher order derivatives of u (see [1, 2]);
- by using finite elements approximations, *i.e.* local functionals like (1.3) defined in suitable spaces of piecewise affine functions (see [10, 12]);
- by considering non-local functionals where the gradient is replaced by finite differences (see [17] and [11] for a numerical implementation).

The last approach was suggested in 1996 by E. De Giorgi, who conjectured the convergence of the family

$$\mathcal{DG}_\varepsilon(u) = \frac{1}{\varepsilon}\int_{\mathbb{R}^n\times\mathbb{R}^n} \arctan\left(\frac{(u(x+\varepsilon\xi)-u(x))^2}{\varepsilon}\right) e^{-|\xi|^2}\,d\xi\,dx,$$

to the Mumford-Shah functional in $\mathbb{R}^n$ (up to some constants), both in the sense of pointwise convergence, and in the sense of Γ-convergence. This conjecture has been proved in [17] by reducing, via an integral-geometric approach, to the simpler family of one-dimensional functionals

$$DG_\varepsilon(u) = \frac{1}{\varepsilon}\int_{\mathbb{R}} \arctan\left(\frac{(u(x+\varepsilon)-u(x))^2}{\varepsilon}\right)\,dx.$$

In this paper we generalize this result. To this end, we consider the family of functionals

$$\mathcal{F}_\varepsilon(u) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \varphi_{\varepsilon|\xi|}\left(\frac{|u(x+\varepsilon\xi)-u(x)|}{\varepsilon|\xi|}\right)\eta(\xi)\,d\xi\,dx, \tag{1.4}$$

where $\{\varphi_\rho\}_{\rho>0}$ is a family of Borel functions, and $\eta\in L^1(\mathbb{R}^n)$.

Our aim is twofold:

- given $\{\varphi_\rho\}$, providing estimates for the Γ-limit of $\{\mathcal{F}_\varepsilon\}$ in terms of $\{\varphi_\rho\}$;
- given a functional $\mathcal{F}$ of the form (1.2), finding $\{\varphi_\rho\}$ such that the family $\{\mathcal{F}_\varepsilon\}$ defined as in (1.4) converges to $\mathcal{F}$.

In particular, if φ and ψ satisfy the usual assumptions in order to have lower semicontinuity of $\mathcal{F}$, and φ is "sectionable" according to Definition 6.1 (*e.g.* $\varphi(r) = |r|^p$ with $p > 1$), then we prove (Theorem 6.3) that there exists $\{\varphi_\rho\}$ such that the following convergence properties are satisfied:

(C1) $\mathcal{F}_\varepsilon(u) \leq \mathcal{F}(u)$ for every $u \in L^1_{loc}(\mathbb{R}^n)$;

(C2) $\{\mathcal{F}_\varepsilon(u)\}$ pointwise converges to $\mathcal{F}(u)$;

(C3) $\mathcal{F}(u)$ is the Γ^--limit of $\{\mathcal{F}_\varepsilon(u)\}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$;

(C4) if $\sup_{\varepsilon>0}\{\mathcal{F}_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty\} < +\infty$, then there exist $\{\varepsilon_j\} \to 0^+$ and $u \in GSBV(\mathbb{R}^n)$ such that $\{u_{\varepsilon_j}\} \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

As in the case of the Mumford-Shah functional, the theory relies almost completely on the study of the simpler family of one-dimensional functionals

$$F_\varepsilon(u) = \int_{\mathbb{R}} \varphi_\varepsilon\left(\frac{|u(x+\varepsilon) - u(x)|}{\varepsilon}\right) dx. \tag{1.5}$$

We point out that pointwise estimates like (C1) are one of the main advantages of this approach. Thanks to such estimates, the passage from the one-dimensional to the n-dimensional case is a simple application of Fatou's lemma and standard integral geometric equalities.

For this reason the finite difference approach is, at the present, the only approach which has been proved to work also with functionals as (1.2), in the case where $\mathcal{F}(u)$ can be finite even if $\mathcal{H}^{n-1}(S_u) = +\infty$ (this happens *e.g.* if $\varphi(r) = r^2$ and $\psi(r) = \sqrt{r}$).

This paper is organized as follows: in § 2 we give notations and preliminaries; in § 3 we study the convergence of the functionals defined in (1.5); in § 4 we consider the general family (1.4) and we prove (C1), (C2), and (C3) under suitable assumptions on $\{\varphi_\rho\}$; in § 5 we consider the compactness property (C4); in § 6 we prove our main approximation result for the functional $\mathcal{F}(u)$ (Theorem 6.3); in § 7 we show some simple examples where the theory developed in this paper applies.

Finally, we would like to thank the referee for carefully reading the manuscript.

2 Preliminaries

In this section we fix notations and we recall basic definitions from the theory of SBV functions and Γ-convergence.

For all $\alpha \in \mathbb{R}$ the integer part of α is denoted by $[\alpha] = \sup\{z \in \mathbb{Z} : z \le \alpha\}$. Given $x, y \in \mathbb{R}^n$, their scalar product is denoted by $\langle x, y\rangle$, and the Euclidean norm of x is denoted by $|x|$. Given $a, b \in \mathbb{R}$, the maximum and the minimum of $\{a, b\}$ are denoted by $a \vee b$ and $a \wedge b$, respectively. Given $A, B \subseteq \mathbb{R}^n$, we write $A \subset\subset B$ if the closure of A is compact and contained in B.

The Lebesgue measure and the $(n-1)$-dimensional Hausdorff measure of a set $B \subseteq \mathbb{R}^n$ are denoted by $|B|$ and $\mathcal{H}^{n-1}(B)$ respectively. The restriction of the measure $\mathcal{H}^{n-1}$ to the set B is denoted by $\mathcal{H}^{n-1}\lfloor_B$. We use standard notations for the Banach spaces $L^p(\mathbb{R}^n)$ and $W^{1,p}(\mathbb{R}^n)$, and for the metrizable spaces $L^p_{loc}(\mathbb{R}^n)$. All the functionals introduced in this paper, and also all the operations of lim, lim inf, lim sup, are intended with range in the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$.

2.1 Special functions of bounded variation

For the general theory of functions with bounded variation we refer to [15, 20]; here we just recall some definitions and some basic results.

Let $\Omega \subseteq \mathbb{R}^n$ be an open set, let $u : \Omega \to \mathbb{R}$ be a measurable function, and let $x \in \Omega$. We denote by $u^+(x)$ and $u^-(x)$, respectively, the upper and lower limit of u at x, defined by

$$\begin{aligned} u^+(x) &:= \inf\left\{t \in \mathbb{R} : \lim_{\rho\to 0^+} \frac{|\{y \in \Omega : \ |x-y| < \rho, \ u(y) > t\}|}{\rho^n} = 0\right\}, \\ u^-(x) &:= \sup\left\{t \in \mathbb{R} : \lim_{\rho\to 0^+} \frac{|\{y \in \Omega : \ |x-y| < \rho, \ u(y) < t\}|}{\rho^n} = 0\right\}. \end{aligned}$$

If $u^+(x) = u^-(x) \in \mathbb{R}$, then x is said to be a *Lebesgue point* of u; in this case, the common value of $u^+(x)$ and $u^-(x)$ is called the *approximate limit* of u at the point x, and is denoted by $\text{ap-}\lim_{y\to x} u(y)$. We denote by S_u the *discontinuity set* of u, *i.e.* the set of all $x \in \Omega$ which are not Lebesgue points of u.

We say that u is a *function of bounded variation* in Ω, and we write $u \in BV(\Omega)$, if $u \in L^1(\Omega)$ and its distributional derivative is a vector-valued

measure Du with finite total variation $|Du|(\Omega)$. We recall that the total variation in Ω can be defined also for every measurable function $v:\Omega\to\mathbb{R}$ by the formula

$$|Dv|(\Omega) \;:=\; \sup\left\{\int_\Omega v\,\mathrm{div}\varphi :\; \varphi\in C_0^\infty(\Omega,\mathbb{R}^n),\;\|\varphi\|_\infty\le 1\right\}. \tag{2.1}$$

If $u\in BV(\Omega)$, then S_u turns out to be countably $(\mathcal{H}^{n-1},n-1)$ rectifiable, *i.e.*

$$S_u = N\cup\bigcup_{i\in\mathbb{N}}K_i,$$

where $\mathcal{H}^{n-1}(N)=0$, and each K_i is a compact set contained in a C^1 hypersurface.

For every $u\in BV(\Omega)$ we have the decomposition $Du=D^a u+D^s u$, where $D^a u$ is absolutely continuous and $D^s u$ is singular with respect to the Lebesgue measure. The density of $D^a u$ with respect to the Lebesgue measure is denoted by ∇u. It turns out that, for almost every $x\in\Omega$, the vector $\nabla u(x)$ is the *approximate gradient* of u, *i.e.*

$$\operatorname*{ap-lim}_{y\to x}\frac{u(y)-u(x)-\langle\nabla u(x),y-x\rangle}{|y-x|}=0.$$

Moreover, we denote the restriction of $D^s u$ to S_u by $D^j u$, and the restriction of $D^s u$ to $\Omega\backslash S_u$ by $D^c u$. With these notations we have the following decomposition:

$$Du=D^a u+D^j u+D^c u.$$

The reader interested in the structure of $D^a u$, $D^j u$, $D^c u$ is referred to [3, 5].

We say that u is a *special function of bounded variation*, and we write $u\in SBV(\Omega)$, if $u\in BV(\Omega)$ and $D^c u=0$. We consider also the larger space $GSBV(\Omega)$, which is composed by all measurable functions $u:\Omega\longrightarrow\mathbb{R}$ whose truncations $u_k=(u\wedge k)\vee(-k)$ belong to $SBV(\Omega')$ for every $k>0$, and every open set $\Omega'\subset\subset\Omega$.

Every $u\in GSBV(\Omega)\cap L^1_{loc}(\Omega)$ has an approximate gradient $\nabla u(x)$ for a.e. $x\in\Omega$, and a countably $(\mathcal{H}^{n-1},n-1)$ rectifiable discontinuity set S_u.

The spaces $SBV(\Omega)$ and $GSBV(\Omega)$ have been introduced by De Giorgi and Ambrosio in [14], and have been studied in [4].

Given two Borel functions $\varphi : [0,+\infty[\to [0,+\infty]$ and $\psi :]0,+\infty] \to [0,+\infty]$, we consider the functional $\mathcal{F}_{\varphi,\psi} : L^1_{loc}(\Omega) \to [0,+\infty]$ defined by

$$\mathcal{F}_{\varphi,\psi}(u,\Omega) = \begin{cases} \displaystyle\int_\Omega \varphi(|\nabla u|)\,dx + \int_{S_u} \psi(|u^+ - u^-|)\,d\mathcal{H}^{n-1} & \text{if } u \in GSBV(\Omega) \\ +\infty & \text{otherwise} \end{cases} \tag{2.2}$$

This functional is "isotropic", in the sense that it is invariant under rigid motions. With an abuse of notation, we denote by $\mathcal{F}_{\varphi,\psi}$ also the non-isotropic functional where the first integral is replaced by the integral of $\varphi(\nabla u)$, where $\varphi : \mathbb{R}^n \to [0,+\infty]$.

In [4] the following semicontinuity result is proved.

Theorem 2.1 *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $\varphi : [0,+\infty[\to [0,+\infty]$ be a non-decreasing convex function such that*

$$\lim_{r\to+\infty} \frac{\varphi(r)}{r} = +\infty, \tag{2.3}$$

and let $\psi :]0,+\infty] \to [0,+\infty]$ be a non-decreasing concave function such that

$$\lim_{r\to 0^+} \frac{\psi(r)}{r} = +\infty. \tag{2.4}$$

Then the functional $\mathcal{F}_{\varphi,\psi}(u,\Omega)$ defined in (2.2) is lower semicontinuous in $L^1_{loc}(\Omega)$. $\square$

2.2 Γ-convergence

Let X be a metric space, let $\{F_i\}$ be a sequence of functions defined in X with values in $\overline{\mathbb{R}}$. Let us set

$$\begin{aligned} \Gamma^-\text{-}\liminf_{i\to\infty} F_i(x) &:= \inf\left\{\liminf_{i\to\infty} F_i(x_i) :\ \{x_i\}\to x\right\}, \\ \Gamma^-\text{-}\limsup_{i\to\infty} F_i(x) &:= \inf\left\{\limsup_{i\to\infty} F_i(x_i) :\ \{x_i\}\to x\right\}. \end{aligned}$$

It turns out that $\Gamma^-\text{-}\liminf_{i\to\infty} F_i(x)$ and $\Gamma^-\text{-}\limsup_{i\to\infty} F_i(x)$ are lower semicontinuous functions. Moreover, the "inf" in the definitions above are actually "min".

If $\Gamma^{-}\text{-}\liminf_{i\to\infty} F_i(x) = \Gamma^{-}\text{-}\limsup_{i\to\infty} F_i(x) = F(x)$ for all $x \in X$, we say that F is the Γ^{-}-limit of $\{F_i\}$, and we write

$$F(x) = \Gamma^{-}\text{-}\lim_{i\to\infty} F_i(x).$$

This means that for every $x \in X$ the following two conditions are satisfied:

(i) if $\{x_i\} \to x$ is any sequence, then $\liminf_{i\to\infty} F_i(x_i) \geq F(x)$;

(ii) there exists a sequence $\{x_i\} \to x$ such that $F(x) \geq \limsup_{i\to\infty} F_i(x_i)$.

The Γ^{-}-limit, when it exists, is unique, and stable under subsequences. The reader interested in variational properties of Γ-convergence is referred to [13].

In general, there is no relation between the Γ^{-}-limit and the pointwise limit. However, if $\{F_i\} \to F$ uniformly on compact subsets of X, then F is also the Γ^{-}-limit of $\{F_i\}$.

A special case is when $F_i(x) = G(x)$ for every $i \in \mathbb{N}$: in this case the Γ^{-}-limit of $\{F_i\}$ is the so called relaxation of G, which we denote by $\overline{G}$. We recall that $\overline{G}$ can also be defined as the supremum of all the lower semicontinuous functions less or equal than G.

Finally, we say that a family $\{F_\varepsilon\}_{\varepsilon>0}$ of functions Γ^{-}-converges to F as $\varepsilon \to 0^+$, if $\{F_{\varepsilon_i}\}$ Γ^{-}-converges to F for every sequence $\{\varepsilon_i\} \to 0^+$.

3 The One-Dimensional Functionals F_ε

In this section we consider a family $\{\varphi_\varepsilon\}_{\varepsilon>0}$ of Borel functions $\varphi_\varepsilon : [0,+\infty[\to [0,+\infty[$, and we study the convergence of the family of functionals

$$F_\varepsilon(u,\Omega) := \int_\Omega \varphi_\varepsilon\left(\frac{|u(x+\varepsilon)-u(x)|}{\varepsilon}\right) dx, \tag{3.1}$$

defined for every $\varepsilon > 0$, $u \in L^1_{loc}(\mathbb{R})$, and every measurable set $\Omega \subseteq \mathbb{R}$, with values in $\mathbb{R} \cup \{+\infty\}$. When $\Omega = \mathbb{R}$, then we simply write $F_\varepsilon(u)$ instead of $F_\varepsilon(u,\mathbb{R})$.

3.1 Statement of the results

We state here all the results which will be proved in this section.

The first one provides an estimate from below for the Γ^{-}-limit of $\{F_\varepsilon\}$.

Theorem 3.1 *Let* $\{\varphi_\varepsilon\}_{\varepsilon>0}$ *be a family of Borel functions such that*

(li1) φ_ε *is continuous and non-decreasing for every* $\varepsilon > 0$*;*

(li2) *for each* $\varepsilon > 0$*, the function* φ_ε *is either convex, or concave, or "convex-concave",* i.e. *there exists* $\overline{r}_\varepsilon > 0$ *such that* φ_ε *is convex in* $[0, \overline{r}_\varepsilon]$ *and concave in* $[\overline{r}_\varepsilon, +\infty[$.

Let us define $\varphi_\star : [0, +\infty[\to [0, +\infty]$ *and* $\psi_\star :]0, +\infty] \to [0, +\infty]$ *by*

$$\varphi_\star(r) := \Gamma^{-}\text{-}\liminf_{\varepsilon\to 0^+} \varphi_\varepsilon(r), \qquad \psi_\star(r) := \Gamma^{-}\text{-}\liminf_{\varepsilon\to 0^+} \varepsilon\varphi_\varepsilon\left(\frac{r}{\varepsilon}\right). \tag{3.2}$$

Then for every $u \in L^1_{loc}(\mathbb{R})$ *we have that*

$$\Gamma^{-}\text{-}\liminf_{\varepsilon\to 0^+} F_\varepsilon(u, \mathbb{R}) \;\geq\; \overline{\mathcal{F}_{\varphi_\star,\psi_\star}}(u, \mathbb{R}),$$

where $\overline{\mathcal{F}_{\varphi_\star,\psi_\star}}$ *is the relaxation of the functional* $\mathcal{F}_{\varphi_\star,\psi_\star}$ *defined as in (2.2).*

The following result provides a pointwise estimate from above for $F_\varepsilon(u, \mathbb{R})$.

Theorem 3.2 *Let* $\{\varphi_\varepsilon\}_{\varepsilon>0}$ *be a family of Borel functions such that*

(li1) φ_ε *is continuous and non-decreasing for every* $\varepsilon > 0$*;*

(Est) *there exist a convex function* $\varphi^\star : [0, +\infty[\to [0, +\infty]$*, and a concave function* $\psi^\star :]0, +\infty] \to [0, +\infty]$ *such that*

$$\varphi_\varepsilon(A + S) \;\leq\; \varphi^\star(A) + \frac{1}{\varepsilon}\,\psi^\star(\varepsilon S),$$

for every $\varepsilon > 0$*,* $A \geq 0$*,* $S > 0$.

Then $F_\varepsilon(u, \mathbb{R}) \leq \overline{\mathcal{F}_{\varphi^\star,\psi^\star}}(u, \mathbb{R})$ *for every* $\varepsilon > 0$*, and every* $u \in L^1_{loc}(\mathbb{R})$.

In many cases, the pointwise limit and the Γ^{-}-limit of $\{F_\varepsilon\}$ are uniquely determined by Theorem 3.1 and Theorem 3.2, as in the following situation.

Corollary 3.3 *Let us assume that the family* $\{\varphi_\varepsilon\}$ *satisfies assumptions (li1), (li2), (Est), and that* $\varphi_\star = \varphi^\star =: \varphi$ *and* $\psi_\star = \psi^\star =: \psi$.

Then

(i) $F_\varepsilon(u,\mathbb{R}) \le \overline{\mathcal{F}_{\varphi,\psi}}(u,\mathbb{R})$ *for every* $u \in L^1_{loc}(\mathbb{R})$, *and every* $\varepsilon > 0$;

(ii) $\{F_\varepsilon(u,\mathbb{R})\}$ *pointwise converges to* $\overline{\mathcal{F}_{\varphi,\psi}}(u,\mathbb{R})$;

(iii) $\overline{\mathcal{F}_{\varphi,\psi}}(u,\mathbb{R})$ *is the* Γ^-*-limit of* $\{F_\varepsilon(u,\mathbb{R})\}$ *in* $L^1_{loc}(\mathbb{R})$.

Remark 3.4 All the results stated above can be generalized word-by-word to the vector valued case $u \in L^1_{loc}(\mathbb{R};\mathbb{R}^k)$.

3.2 Estimates from below

In this subsection we prove Theorem 3.1. The strategy of the proof follows the argument used in [17] in the case of the Mumford-Shah functional.

In order to avoid a cumbersome notation, we extend to $[0,+\infty]$ the function $\psi_\star$, defined in (3.2), by setting $\psi_\star(0)=0$. Moreover, for every $\alpha \ge 0$, $\beta > 0$, we define

$$\lambda(\alpha,\beta) \;:=\; \min\left\{\beta\varphi_\star\left(\frac{\alpha-l}{\beta}\right) + \psi_\star(l) \,:\; 0 \le l \le \alpha\right\}. \tag{3.3}$$

Since $\varphi_\star$ and the extension of $\psi_\star$ are lower semicontinuous on $[0,+\infty[$, it follows that λ is well defined and lower semicontinuous. Moreover, since we can always set $l=0$ in (3.3), it turns out that

$$\lambda(\alpha,\beta) \;\le\; \beta\,\varphi_\star\left(\frac{\alpha}{\beta}\right). \tag{3.4}$$

Now we state and prove three technical lemmata. The first one is a "discretization" of Theorem 3.1.

Lemma 3.5 *Let* $\{\varphi_\varepsilon\}_{\varepsilon>0}$ *be a family of Borel functions satisfying (li1), (li2), and let* $\alpha \ge 0$ *and* $\beta > 0$. *Then for every* $\varepsilon \in]0,\beta]$, *there exists*

$$\Theta(\varepsilon,\alpha,\beta) := \min\left\{\sum_{i=1}^{N_\varepsilon} \varepsilon\varphi_\varepsilon\left(\frac{|x_i|}{\varepsilon}\right) \,:\; \sum_{i=1}^{N_\varepsilon}|x_i| \ge \alpha,\; N_\varepsilon = \left[\frac{\beta}{\varepsilon}\right]\right\}. \tag{3.5}$$

Moreover,

$$\liminf_{\varepsilon\to0^+} \Theta(\varepsilon,\alpha,\beta) \;\ge\; \lambda(\alpha,\beta), \tag{3.6}$$

where λ *is the function defined in (3.3).*

PROOF. The minimum problem (3.5) has at least one solution, since by (li1) we can restrict to the compact set

$$\left\{(x_1, x_2, \dots, x_{N_\varepsilon}) \in [0,\alpha]^{N_\varepsilon} :\ \sum_{i=1}^{N_\varepsilon} x_i = \alpha\right\}.$$

Therefore the function $\Theta(\varepsilon,\alpha,\beta)$ is well defined.

Now let $\{\varepsilon_n\} \to 0^+$ be a sequence such that

$$\liminf_{\varepsilon\to 0^+} \Theta(\varepsilon,\alpha,\beta) \ = \ \lim_{n\to\infty} \Theta(\varepsilon_n,\alpha,\beta),$$

and, for all ε_n, let $x_{n,1} \geq x_{n,2} \geq \dots \geq x_{n,N_{\varepsilon_n}}$ be a minimizer for (3.5).

Since by (li2) the function $r \mapsto \varepsilon_n \varphi_{\varepsilon_n}(r/\varepsilon_n)$ is convex in $[0, \varepsilon_n \overline{r}_{\varepsilon_n}]$ and concave in $[\varepsilon_n \overline{r}_{\varepsilon_n}, +\infty[$ (with obvious modifications if φ_ε is always convex or always concave), it follows that only $x_{n,1}$ can be greater than $\varepsilon_n \overline{r}_{\varepsilon_n}$, and all the $x_{n,i}$'s in the convexity zone are equal (this is true if φ_ε is strictly convex in $[0, \overline{r}_\varepsilon]$; however, if φ_ε has a flat zone in $[0, \overline{r}_\varepsilon]$, then there exists at least one minimizer with the property that all the $x_{n,i}$'s in the convexity zone are equal, and so we can work with this minimizer without loss of generality). Therefore, there are only two possibilities:

(P1) $x_{n,1} = \dots = x_{n,N_{\varepsilon_n}} = \alpha/N_{\varepsilon_n}$, and in this case

$$\Theta(\varepsilon_n,\alpha,\beta) \ = \ \varepsilon_n N_{\varepsilon_n} \varphi_{\varepsilon_n}\left(\frac{\alpha}{\varepsilon_n N_{\varepsilon_n}}\right); \tag{3.7}$$

(P2) $x_{n,1} \geq \varepsilon_n \overline{r}_{\varepsilon_n}$ and $x_{n,2} = \dots = x_{n,N_{\varepsilon_n}} = (\alpha - x_{n,1})/(N_{\varepsilon_n} - 1)$. In this case

$$\Theta(\varepsilon_n,\alpha,\beta) \ = \ \varepsilon_n \varphi_{\varepsilon_n}\left(\frac{x_{n,1}}{\varepsilon_n}\right) + \varepsilon_n (N_{\varepsilon_n} - 1)\, \varphi_{\varepsilon_n}\left(\frac{\alpha - x_{n,1}}{\varepsilon_n (N_{\varepsilon_n} - 1)}\right). \tag{3.8}$$

Up to subsequences, we can suppose that either (P1) or (P2) holds true for all $n \in \mathbb{N}$. In the first case, observing that $\{\varepsilon_n N_{\varepsilon_n}\} \to \beta$ and using the definition of $\varphi_\star$, passing to the limit in (3.7) we have that

$$\liminf_{n\to\infty} \Theta(\varepsilon_n,\alpha,\beta) \ \geq \ \beta\, \varphi_\star\left(\frac{\alpha}{\beta}\right) \ \geq \ \lambda(\alpha,\beta).$$

In the second case, up to subsequences, we can assume that there exists

$$l = \lim_{n\to\infty} x_{n,1} \in [0,\alpha].$$

By the definition of $\varphi_\star$, $\psi_\star$, and λ, passing to the limit in (3.8) we obtain that

$$\liminf_{n\to\infty} \Theta(\varepsilon_n,\alpha,\beta) \;\geq\; \psi_\star(l) + \beta\varphi_\star\left(\frac{\alpha - l}{\beta}\right) \;\geq\; \lambda(\alpha,\beta).$$

In both cases, inequality (3.6) is proved. $\square$

The second lemma is a "localization" of Theorem 3.1.

Lemma 3.6 *Let $I = [a,b]$ be an interval, let $\{u_\varepsilon\} \subseteq L^1_{loc}(\mathbb{R})$, and let $u \in L^1_{loc}(\mathbb{R})$. Let us assume that*

(i) $u_\varepsilon \to u$ *in* $L^1_{loc}(\mathbb{R})$*;*

(ii) a *and* b *are Lebesgue points of* u*.*

Then

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, I) \;\geq\; \lambda(|u(b)-u(a)|, b-a). \tag{3.9}$$

Proof. Let $\{\varepsilon_n\} \to 0^+$ be a sequence such that

$$\liminf_{\varepsilon\to 0^+} F_\varepsilon(u_\varepsilon, I) \;=\; \lim_{n\to\infty} F_{\varepsilon_n}(u_{\varepsilon_n}, I).$$

Up to subsequences, we can assume that

$$u_{\varepsilon_n}(x) \longrightarrow u(x) \qquad \text{for a.e. } x \in I.$$

Now, let us set $J := |u(b) - u(a)|$. If $J = 0$, then there is nothing to prove. Otherwise, let us fix $\eta \in]0, J]$, let us set $N_{\varepsilon_n} = [(b-a)/\varepsilon_n]$, and let us define

$$C_n = \left\{ x \in [a, a+\varepsilon_n] : \sum_{k=1}^{N_{\varepsilon_n}} |u_{\varepsilon_n}(x + k\varepsilon_n) - u_{\varepsilon_n}(x + (k-1)\varepsilon_n)| \geq J - \eta \right\}.$$

Using assumption (ii), it can be proved (for the technical details see Step 2 in the proof of Lemma 3.2 in [17]) that

$$\lim_{n\to\infty} \frac{|C_n|}{\varepsilon_n} \;=\; 1. \tag{3.10}$$

By the definition of C_n we obtain that

$$\begin{aligned} F_{\varepsilon_n}(u_{\varepsilon_n}, I) &\geq F_{\varepsilon_n}(u_{\varepsilon_n}, [a, a+\varepsilon_n N_{\varepsilon_n}]) \\ &= \int_a^{a+\varepsilon_n N_{\varepsilon_n}} \varphi_{\varepsilon_n}\left(\frac{|u_{\varepsilon_n}(x+\varepsilon_n) - u_{\varepsilon_n}(x)|}{\varepsilon_n}\right) dx \\ &= \frac{1}{\varepsilon_n}\int_a^{a+\varepsilon_n} \sum_{k=1}^{N_{\varepsilon_n}} \varepsilon_n \varphi_{\varepsilon_n}\left(\frac{|u_{\varepsilon_n}(x+k\varepsilon_n) - u_{\varepsilon_n}(x+(k-1)\varepsilon_n)|}{\varepsilon_n}\right) dx \\ &\geq \frac{|C_n|}{\varepsilon_n}\,\Theta(\varepsilon_n, J-\eta, b-a) \end{aligned}$$

where Θ is the function defined in (3.5). Applying Lemma 3.5 with $\alpha = J-\eta$ and $\beta = b-a$, and using (3.10), we conclude that

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_{\varepsilon_n}, I) \;\geq\; \liminf_{n\to\infty} \frac{|C_n|}{\varepsilon_n}\,\Theta(\varepsilon_n, J-\eta, b-a) \;\geq\; \lambda(J-\eta, b-a).$$

Since λ is lower semicontinuous, and η is arbitrary, (3.9) is proved. □

The third Lemma states a general property of $L^p(\mathbb{R})$ spaces (for a proof, see Lemma 3.3 in [17]).

Lemma 3.7 *Let $u \in L^\infty(\mathbb{R})$. Then there exists $a \in \mathbb{R}$ such that*

(i) *$a+q$ is a Lebesgue point of u for every rational number q;*

(ii) *every sequence $\{u_n\} \subset L^\infty(\mathbb{R})$ which satisfies the following two conditions*

- *$u_n(a+\frac{z}{n}) = u(a+\frac{z}{n})$ for all $z \in \mathbb{Z}$,*
- *if $x \in [a+\frac{z}{n}, a+\frac{(z+1)}{n}]$, then $u_n(x)$ belongs to the interval with endpoints $u(a+\frac{z}{n})$ and $u(a+\frac{(z+1)}{n})$,*

has a subsequence converging to u in $L^1_{loc}(\mathbb{R})$. □

Proof of Theorem 3.1. Let us set for simplicity $F_\star := \mathcal{F}_{\varphi_\star,\psi_\star}$. We have to show that

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n) \geq \overline{F_\star}(u)$$

for every $u \in L^1_{loc}(\mathbb{R})$, every $\{\varepsilon_n\} \to 0^+$, and every sequence $\{u_n\} \to u$ in $L^1_{loc}(\mathbb{R})$. Let us begin with the case where $u \in L^\infty(\mathbb{R})$, $\{u_n\} \subseteq L^\infty(\mathbb{R})$, and $\|u_n\|_\infty \leq \|u\|_\infty$.

Our strategy is to construct a sequence $\{v_j\} \subseteq GSBV(\mathbb{R}) \cap L^1_{loc}(\mathbb{R})$ such that

$$\{v_j\} \to u \qquad \text{in } L^1_{loc}(\mathbb{R}), \tag{3.11}$$

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n) \geq F_\star(v_j) \qquad \forall j \in \mathbb{N}. \tag{3.12}$$

Let us assume that $a \in [0,1]$ satisfies conditions (i) and (ii) of Lemma 3.7. For all $z \in \mathbb{Z}$ and $j \in \mathbb{N}$, let $\beta = 1/j$, let I_j^z be the interval $[a+z\beta, a+(z+1)\beta]$, let J_j^z be the increment $|u(a+(z+1)\beta) - u(a+z\beta)|$, and let l_j^z be such that

$$\lambda(J_j^z, \beta) \;=\; \beta\varphi_\star\left(\frac{J_j^z - l_j^z}{\beta}\right) + \psi_\star(l_j^z).$$

Now we define v_j on every interval I_j^z as the piecewise affine function which

- coincides with u at the endpoints of I_j^z;
- has constant (approximate) gradient in the interval, with $|\nabla v_j(x)| = (J_j^z - l_j^z)/\beta$;
- has a jump of height l_j^z in the medium point of the interval (of course if $l_j^z = 0$, then no jump point is necessary).

It is easy to check that the functions v_j satisfy both assumptions of (ii) of Lemma 3.7; hence, up to subsequences, (3.11) holds. Moreover, v_j belongs to $GSBV(\mathbb{R}) \cap L^1_{loc}(\mathbb{R})$, and for all $z \in \mathbb{Z}$, $j \in \mathbb{N}$, we have that

$$F_\star(v_j, I_j^z) \;=\; \lambda(J_j^z, \beta),$$

hence, by Lemma 3.6 applied in the interval I_j^z,

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n, I_j^z) \;\geq\; \lambda(J_j^z, \beta) \;=\; F_\star(v_j, I_j^z).$$

Summing over all $z \in \mathbb{Z}$ and using Fatou's Lemma for series, it follows that (3.12) holds true, and this completes the proof in the case $u \in L^\infty(\mathbb{R})$.

In the general case $u \in L^1_{loc}(\mathbb{R})$, let us denote by T_k the truncation operator $T_k v = (v \vee k) \wedge k$. For every $k > 0$ we have that $\{T_k u_n\} \to T_k u$ as $n \to +\infty$.

Moreover, since φ_ε is non-decreasing we have that $F_{\varepsilon_n}(u_n,\mathbb{R}) \geq F_{\varepsilon_n}(T_k u_n,\mathbb{R})$ so that

$$\liminf_{n\to\infty} F_{\varepsilon_n}(u_n,\mathbb{R}) \;\geq\; \liminf_{n\to\infty} F_{\varepsilon_n}(T_k u_n,\mathbb{R}) \;\geq\; F_\star(T_k u,\mathbb{R}), \tag{3.13}$$

where the last inequality follows from the L^∞ case proved above.

Since $\{T_k u\} \to u$ in $L^1_{loc}(\mathbb{R})$ as $k \to +\infty$, then the conclusion follows letting $k \to +\infty$, due to (3.13) and the lower semi-continuity of $F_\star$. □

3.3 Pointwise estimates

In this subsection we prove Theorem 3.2 and Corollary 3.3.

Proof of Theorem 3.2. Since F_ε is lower semicontinuous in $L^1_{loc}(\mathbb{R})$ (by Fatou's Lemma), then it is enough to prove that $F_\varepsilon(u) \leq \mathcal{F}_{\varphi^\star,\psi^\star}(u)$. To this end, we can of course assume that $\mathcal{F}_{\varphi^\star,\psi^\star}(u) < +\infty$, hence $u \in SBV(J)$ for every $J \subset\subset \mathbb{R}$. In this case let us set, for every $x \in \mathbb{R}$,

$$\begin{aligned} A_\varepsilon(x) &:= |D^a u|([x,x+\varepsilon]) = \int_0^\varepsilon |\nabla u(x+\tau)|\, d\tau; \\ S_\varepsilon(x) &:= \left|D^j u\right|([x,x+\varepsilon]) = \int_x^{x+\varepsilon} \left|u^+(\tau)-u^-(\tau)\right|\, d\mathcal{H}^0\lfloor_{S_u}(\tau). \end{aligned}$$

Since $|u(x+\varepsilon)-u(x)| \leq A_\varepsilon(x)+S_\varepsilon(x)$ for a.e. $x \in \mathbb{R}$, by (Est) we have that

$$\varphi_\varepsilon\left(\frac{|u(x+\varepsilon)-u(x)|}{\varepsilon}\right) \;\leq\; \varphi^\star\left(\frac{A_\varepsilon(x)}{\varepsilon}\right) + \frac{1}{\varepsilon}\,\psi^\star\left(S_\varepsilon(x)\right). \tag{3.14}$$

Now let us estimate separately the integral of the two summands. Since $\varphi^\star$ is convex, by Jensen's inequality we have that

$$\begin{aligned} \int_{\mathbb{R}} \varphi^\star\left(\frac{A_\varepsilon(x)}{\varepsilon}\right) dx &= \int_{\mathbb{R}} \varphi^\star\left(\frac{1}{\varepsilon}\int_0^\varepsilon |\nabla u(x+\tau)|\, d\tau\right) dx \\ &\leq \int_{\mathbb{R}} \frac{1}{\varepsilon}\int_0^\varepsilon \varphi^\star\left(|\nabla u(x+\tau)|\right) d\tau\, dx \\ &= \frac{1}{\varepsilon}\int_0^\varepsilon d\tau \int_{\mathbb{R}} \varphi^\star\left(|\nabla u(x+\tau)|\right) dx \\ &= \int_{\mathbb{R}} \varphi^\star\left(|\nabla u(x)|\right) dx \end{aligned} \tag{3.15}$$

Since $\psi^\star$ is subadditive, then

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{\mathbb{R}} \psi^\star\left(S_\varepsilon(x)\right) dx &= \frac{1}{\varepsilon}\int_{\mathbb{R}} \psi^\star\left(\int_x^{x+\varepsilon} \left|u^+(\tau)-u^-(\tau)\right| \, d\mathcal{H}^0\lfloor_{S_u}(\tau)\right) dx \\
&\leq \frac{1}{\varepsilon}\int_{\mathbb{R}} \left(\int_x^{x+\varepsilon} \psi^\star\left(\left|u^+(\tau)-u^-(\tau)\right|\right) \, d\mathcal{H}^0\lfloor_{S_u}(\tau)\right) dx \\
&= \frac{1}{\varepsilon}\int_{\mathbb{R}} \psi^\star\left(\left|u^+(\tau)-u^-(\tau)\right|\right) \, d\mathcal{H}^0\lfloor_{S_u}(\tau) \int_{\tau-\varepsilon}^{\tau} dx \\
&= \int_{S_u} \psi^\star\left(\left|u^+(\tau)-u^-(\tau)\right|\right) \, d\mathcal{H}^0(\tau). \qquad (3.16)
\end{aligned}$$

By (3.14), (3.15), and (3.16), thesis is proved. $\square$

Proof of Corollary 3.3. The family $\{\varphi_\varepsilon\}$ satisfies assumptions (li1) and (Est) of Theorem 3.2 with $\varphi = \varphi^\star$ and $\psi = \psi^\star$. This proves statement (i), and in particular

$$\limsup_{\varepsilon\to 0^+} F_\varepsilon(u) \;\leq\; \overline{\mathcal{F}_{\varphi,\psi}}(u), \qquad \forall\, u \in L^1_{loc}(\mathbb{R}). \qquad (3.17)$$

Moreover, $\{\varphi_\varepsilon\}$ satisfies assumptions (li1) and (li2) of Theorem 3.1. Since $\varphi = \varphi_\star$ and $\psi = \psi_\star$, it follows that

$$\Gamma^- \text{-} \liminf_{\varepsilon\to 0^+} F_\varepsilon(u) \;\geq\; \overline{\mathcal{F}_{\varphi,\psi}}(u), \qquad \forall\, u \in L^1_{loc}(\mathbb{R}). \qquad (3.18)$$

By (3.17) and (3.18), statements (ii) and (iii) follow. $\square$

4 The general family $\mathcal{F}_\varepsilon$

In this section we consider a family $\{\varphi_\varepsilon\}_{\varepsilon>0}$ of Borel functions as in § 3, and a non-negative function $\eta \in L^1(\mathbb{R}^n)$. We study the convergence of the family of functionals

$$\mathcal{F}_\varepsilon(u,\mathbb{R}^n) = \int_{\mathbb{R}^n\times\mathbb{R}^n} \varphi_{\varepsilon|\xi|}\left(\frac{|u(x+\varepsilon\xi)-u(x)|}{\varepsilon|\xi|}\right)\eta(\xi)\, d\xi\, dx, \qquad (4.1)$$

defined for every $\varepsilon > 0$, and every $u \in L^1_{loc}(\mathbb{R}^n)$.

The advantage of $\{\mathcal{F}_\varepsilon\}$ with respect to the family $\{F_\varepsilon\}$ introduced in § 3 is twofold:

- it can be defined in every space dimension;
- it fulfills the compactness properties stated in § 5 (the family $\{F_\varepsilon\}$, on the contrary, satisfies no compactness properties).

However, the results of § 3 are a fundamental tool in the study of the convergence of $\{\mathcal{F}_\varepsilon\}$, due to integral geometric techniques. To this end, we introduce the functionals

$$F_{\varepsilon,\xi}(u,\mathbb{R}^n) = \int_{\mathbb{R}^n} \varphi_{\varepsilon|\xi|}\left(\frac{|u(x+\varepsilon\xi)-u(x)|}{\varepsilon|\xi|}\right) dx, \tag{4.2}$$

defined for every $\varepsilon > 0$, $\xi \in \mathbb{R}^n \setminus \{0\}$, $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. With this notation

$$\mathcal{F}_\varepsilon(u,\mathbb{R}^n) = \int_{\mathbb{R}^n} F_{\varepsilon,\xi}(u,\mathbb{R}^n)\,\eta(\xi)\,d\xi \qquad \forall\, u \in L^1_{\mathrm{loc}}(\mathbb{R}^n). \tag{4.3}$$

Now let $\xi \in \mathbb{R}^n \setminus \{0\}$, and let $\langle\xi\rangle^\perp = \{z \in \mathbb{R}^n : \ \langle \xi, z\rangle = 0\}$ be the orthogonal space to ξ. For every $y \in \langle\xi\rangle^\perp$ let us consider the function $u_{\xi,y} : \mathbb{R} \to \mathbb{R}$ defined by

$$u_{\xi,y}(t) = u\left(y + t\frac{\xi}{|\xi|}\right), \qquad \forall\, t \in \mathbb{R}. \tag{4.4}$$

With the substitution $x = y + t\xi/|\xi|$, relation (4.2) can be rewritten as

$$\begin{aligned} F_{\varepsilon,\xi}(u,\mathbb{R}^n) &= \int_{\langle\xi\rangle^\perp}\int_{\mathbb{R}} \varphi_{\varepsilon|\xi|}\left(\frac{|u(y+t\xi/|\xi|+\varepsilon\xi)-u(y+t\xi/|\xi|)|}{\varepsilon|\xi|}\right) dt\,dy \\ &= \int_{\langle\xi\rangle^\perp}\int_{\mathbb{R}} \varphi_{\varepsilon|\xi|}\left(\frac{|u_{\xi,y}(t+\varepsilon|\xi|)-u_{\xi,y}(t)|}{\varepsilon|\xi|}\right) dt\,dy \\ &= \int_{\langle\xi\rangle^\perp} F_{\varepsilon|\xi|}(u_{\xi,y},\mathbb{R})\,dy \end{aligned} \tag{4.5}$$

where $\{F_\varepsilon\}$ is the family defined in (3.1).

Thanks to (4.3) and (4.5), the functional $\mathcal{F}_\varepsilon(u,\mathbb{R}^n)$ can be written in terms of the one-dimensional sections of u.

We now need the following result about one-dimensional sections of $GSBV$ functions.

Lemma 4.1 *Let φ and ψ be as in the lower semicontinuity theorem 2.1.*

(i) *Let* $u \in GSBV(\mathbb{R}^n)$*. Then for all* $\xi \in \mathbb{R}^n$ *we have that* $u_{\xi,y} \in GSBV(\mathbb{R})$ *for a.e.* $y \in \langle\xi\rangle^\perp$*, and moreover*

$$\begin{aligned} \nabla u_{\xi,y}(t) &= \langle \nabla u(y + t\xi/|\xi|), \xi/|\xi|\rangle, \qquad \text{for a.e. } t \in \mathbb{R}; \qquad (4.6) \\ S_{u_{\xi,y}} &= \{t \in \mathbb{R} : \ y + t\xi/|\xi| \in S_u\}; \qquad (4.7) \end{aligned}$$

$$u^+_{\xi,y}(t) = u^+\left(y + t\xi/|\xi|\right), \quad u^-_{\xi,y}(t) = u^-\left(y + t\xi/|\xi|\right) \quad \forall\, t \in \mathbb{R}. \qquad (4.8)$$

(ii) *Vice-versa: let* $u \in L^1_{loc}(\mathbb{R}^n)$*, and let* $\{\xi_1, \ldots, \xi_n\} \subseteq \mathbb{R}^n$ *be a set of linearly independent vectors. If*

$$\int_{\langle\xi_i\rangle^\perp} \mathcal{F}_{\varphi,\psi}(u_{\xi_i,y}, \mathbb{R})\, dy < +\infty \qquad (4.9)$$

for all $i \in \{1, \ldots, n\}$*, then* $u \in GSBV(\mathbb{R}^n)$*.*

(iii) *If* $\eta \in L^1(\mathbb{R}^n)$ *is a non-negative non-zero function, then for every* $u \in L^1_{loc}(\mathbb{R}^n)$ *we have that*

$$\int_{\mathbb{R}^n} \left(\int_{\langle\xi\rangle^\perp} \mathcal{F}_{\varphi,\psi}(u_{\xi,y}, \mathbb{R})\, dy \right) \eta(\xi)\, d\xi \ = \ \omega\, \mathcal{F}_{S\varphi,\psi}(u, \mathbb{R}^n), \qquad (4.10)$$

where $\omega := \|\eta\|_{L^1(\mathbb{R})}$*, and*

$$(S\varphi)(z) := \frac{1}{\omega} \int_{\mathbb{R}^n} \varphi\left(|\langle z, \xi/|\xi|\rangle|\right) \eta(\xi)\, d\xi. \qquad (4.11)$$

Proof. Statements (i) and (ii) follow from [3, Theorem 3.3].

In order to prove (iii), let us assume first that $u \in GSBV(\mathbb{R}^n) \cap L^1_{loc}(\mathbb{R}^n)$. In this case by (4.6), (4.7), and (4.8) we have that

$$\begin{aligned} \int_{\langle\xi\rangle^\perp} \mathcal{F}_{\varphi,\psi}(u_{\xi,y}, \mathbb{R})\, dy \ &= \ \int_{\langle\xi\rangle^\perp} \int_{\mathbb{R}} \varphi\left(\left| \langle \nabla u \left(y + t\frac{\xi}{|\xi|} \right), \frac{\xi}{|\xi|} \rangle \right| \right) dt\, dy + \\ &+ \int_{\langle\xi\rangle^\perp} \int_{S_{u_{\xi,y}}} \psi\left(\left| u^+_{\xi,y}(t) - u^-_{\xi,y}(t) \right| \right) d\mathcal{H}^0(t)\, dy \\ &= \ \int_{\mathbb{R}^n} \varphi\left(|\langle \nabla u(x), \xi/|\xi|\rangle|\right) dx + \int_{S_u} \psi\left(\left| u^+(y) - u^-(y) \right| \right) d\mathcal{H}^{n-1}(y), \end{aligned}$$

where the last equality follows from the substitution $x = y + t\xi/|\xi|$ for the first summand, and from [16, Theorem 3.2.26] for the second summand. Multiplying this equality by $\eta(\xi)$, and integrating in ξ over $\mathbb{R}^n$, we prove (4.10) in this case.

If $u \in L^1_{loc}(\mathbb{R}^n) \setminus GSBV(\mathbb{R}^n)$, then necessarily

$$\int_{\langle\xi\rangle^\perp} \mathcal{F}_{\varphi,\psi}(u_{\xi,y}, \mathbb{R})\, dy \;=\; +\infty \qquad \text{for a.e. } \xi \in \mathbb{R}^n,$$

hence both sides of (4.10) are equal to $+\infty$. Indeed, if this is not the case, then we can find a set of linearly independent vectors $\{\xi_1, \ldots, \xi_n\}$ such that (4.9) holds true for every $i \in \{1, \ldots, n\}$, hence $u \in GSBV(\mathbb{R}^n)$ (by statement (ii)), which is impossible. $\Box$

Remark 4.2 If $\eta(\xi)$ is radial, *i.e.* depends only on $|\xi|$, then the function $(S\varphi)\,(z)$ defined in (4.11) depends only on $|z|$. In particular, if $\varphi(z) = |z|^p$ (and η is radial), then $(S\varphi)\,(z) = c_{p,n}\omega^{-1}|z|^p$, where

$$c_{p,n} := \int_{S^{n-1}} |\langle v, e_1\rangle|^p \; d\mathcal{H}^{n-1}(v), \qquad e_1 := (1, 0, \ldots, 0). \tag{4.12}$$

Combining the results of § 3 with equalities (4.3), (4.5), and (4.10) we can study the convergence of $\{\mathcal{F}_\varepsilon\}$. For shortness' shake, we only give the following result.

Theorem 4.3 *Let φ and ψ be as in the lower semicontinuity Theorem 2.1, and let $\eta \in L^1(\mathbb{R}^n)$ be a non-negative non-zero function. Let $\{\varphi_\varepsilon\}$ be a family of Borel functions satisfying assumptions (li1), (li2), and (Est) (cf. Theorem 3.1 and Theorem 3.2) with $\varphi = \varphi_\star = \varphi^\star$ and $\psi = \psi_\star = \psi^\star$. Finally, let $\omega := \|\eta\|_{L^1(\mathbb{R}^n)}$, and let $S\varphi$ be the function defined in (4.11).*

Then

(i) $\mathcal{F}_\varepsilon(u, \mathbb{R}^n) \leq \omega\, \mathcal{F}_{S\varphi,\psi}(u, \mathbb{R}^n)$ *for every* $u \in L^1_{loc}(\mathbb{R}^n)$*, and every* $\varepsilon > 0$*;*

(ii) $\{\mathcal{F}_\varepsilon(u, \mathbb{R}^n)\}$ *pointwise converges to* $\omega\, \mathcal{F}_{S\varphi,\psi}(u, \mathbb{R}^n)$*;*

(iii) $\omega\, \mathcal{F}_{S\varphi,\psi}(u, \mathbb{R}^n)$ *is the* Γ^-*-limit of* $\{\mathcal{F}_\varepsilon(u, \mathbb{R}^n)\}$ *in* $L^1_{loc}(\mathbb{R}^n)$*.*

PROOF. Let us prove statement (i). By (4.3), (4.5), and (4.10) we have that

$$\begin{aligned}
\mathcal{F}_\varepsilon(u,\mathbb{R}^n) &= \int_{\mathbb{R}^n} F_{\varepsilon,\xi}(u,\mathbb{R}^n)\,\eta(\xi)\,d\xi \\
&= \int_{\mathbb{R}^n}\left(\int_{\langle\xi\rangle^\perp} F_{\varepsilon|\xi|}(u_{\xi,y},\mathbb{R})\,dy\right)\eta(\xi)\,d\xi \\
&\leq \int_{\mathbb{R}^n}\left(\int_{\langle\xi\rangle^\perp} \mathcal{F}_{\varphi,\psi}(u_{\xi,y},\mathbb{R})\,dy\right)\eta(\xi)\,d\xi \;=\; \omega\,\mathcal{F}_{S\varphi,\psi}(u,\mathbb{R}^n).
\end{aligned}$$

In order to complete the proof, it remains to show that

$$\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n,\mathbb{R}^n) \;\geq\; \omega\,\mathcal{F}_{S\varphi,\psi}(u,\mathbb{R}^n)$$

for every sequence $\{\varepsilon_n\}\to 0^+$, and every sequence $\{u_n\}\to u$ in $L^1_{loc}(\mathbb{R}^n)$. Since

$$\{(u_n)_{\xi,y}\} \longrightarrow u_{\xi,y} \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{R})$$

for a.e. $y\in\langle\xi\rangle^\perp$, by (4.3), (4.5), and Fatou's Lemma we have that:

$$\begin{aligned}
\liminf_{n\to\infty} \mathcal{F}_{\varepsilon_n}(u_n,\mathbb{R}^n) &= \liminf_{n\to\infty}\int_{\mathbb{R}^n} F_{\varepsilon_n,\xi}(u_n,\mathbb{R}^n)\,\eta(\xi)\,d\xi \\
&= \liminf_{n\to\infty}\int_{\mathbb{R}^n}\left(\int_{\langle\xi\rangle^\perp} F_{\varepsilon_n|\xi|}((u_n)_{\xi,y},\mathbb{R})\,dy\right)\eta(\xi)\,d\xi \\
&\geq \int_{\mathbb{R}^n}\left(\int_{\langle\xi\rangle^\perp} \liminf_{n\to\infty} F_{\varepsilon_n|\xi|}((u_n)_{\xi,y},\mathbb{R})\,dy\right)\eta(\xi)\,d\xi \\
&\geq \int_{\mathbb{R}^n}\left(\int_{\langle\xi\rangle^\perp} \mathcal{F}_{\varphi,\psi}(u_{\xi,y},\mathbb{R})\,dy\right)\eta(\xi)\,d\xi \;=\; \omega\,\mathcal{F}_{S\varphi,\psi}(u,\mathbb{R}^n).
\end{aligned}$$

This completes the proof. $\square$

Remark 4.4 If in Theorem 4.3 we drop the assumption that φ and ψ satisfy (2.3) and (2.4), then the pointwise limit, the Γ^--limit, and an upper estimate for $\mathcal{F}_\varepsilon(u,\mathbb{R}^n)$ are given by the functional

$$\tilde{\mathcal{F}}(u,\mathbb{R}^n) := \int_{\mathbb{R}^n}\left(\int_{\langle\xi\rangle^\perp} \overline{\mathcal{F}_{\varphi,\psi}}(u_{\xi,y},\mathbb{R})\,dy\right)\eta(\xi)\,d\xi, \tag{4.13}$$

where $\overline{\mathcal{F}_{\varphi,\psi}}$ is the relaxation of $\mathcal{F}_{\varphi,\psi}$ in the one-dimensional case.

Remark 4.5 All the results of this section are true also in the particular case where $n = 1$. In this case $\langle \xi \rangle^{\perp} = \{0\}$ for every $\xi \in \mathbb{R} \setminus \{0\}$, and therefore many formulas containing integrations over $\langle \xi \rangle^{\perp}$ may be simplified. Moreover (4.11) reduces to

$$(S\varphi)(z) \;=\; \frac{1}{\omega} \int_{\mathbb{R}} \varphi(|z|)\eta(\xi)\, d\xi \;=\; \varphi(|z|).$$

Remark 4.6 All the results of this section (and in particular Theorem 4.3) can be generalized to an arbitrary open set $\Omega \subseteq \mathbb{R}^n$. In this case the natural generalization of (4.1) are the functionals

$$\mathcal{F}_{\varepsilon}(u, \Omega) \;:=\; \frac{1}{\varepsilon^n} \int_{\mathrm{vis}(\Omega)} \varphi_{|y-x|} \left(\frac{|u(y) - u(x)|}{|y - x|} \right) \eta \left(\frac{y - x}{\varepsilon} \right) dy\, dx, \qquad (4.14)$$

where

$$\mathrm{vis}(\Omega) := \{(x, y) \in \Omega \times \Omega :\ \forall\, t \in [0, 1],\ tx + (1 - t)y \in \Omega\}$$

is the set of all pairs in $\Omega \times \Omega$ which "see each other".

If $\Omega = \mathbb{R}^n$, then (4.1) can be written as (4.14) with the substitution $x + \varepsilon\xi = y$. The restriction of the integration to $\mathrm{vis}(\Omega)$, instead of $\Omega \times \Omega$, makes this construction to work on every open set Ω, without any assumption on the regularity of the boundary (see the discussion in [17, section 7]).

5 Compactness

In this section we prove the following compactness result.

Theorem 5.1 *Let $\{\varphi_{\varepsilon}\}$ be a family of Borel functions such that*

(Cpt1) *for every $M > 0$ there exist $H_M > 0$ and $K_M \geq 0$ such that*

$$\varphi_{\varepsilon}(r) \;\geq\; H_M\, r - K_M \qquad \forall\, r \in [0, M/\varepsilon];$$

(Cpt2) *φ_{ε} is nondecreasing for every $\varepsilon > 0$, and*

$$\varphi_{(k+1)\varepsilon} \left(\frac{A + B}{k + 1} \right) \;\leq\; \frac{1}{k + 1}\, \varphi_{\varepsilon}(A) + \frac{k}{k + 1}\, \varphi_{k\varepsilon} \left(\frac{B}{k} \right)$$

for every $A \geq 0$, $B \geq 0$, $\varepsilon > 0$, $k \in \mathbb{N} \setminus \{0\}$.

Let $\eta \in L^1(\mathbb{R}^n)$ be a non-negative function such that $\{\xi \in \mathbb{R}^n : \ \eta(\xi) > c\}$ has non-empty interior for some $c > 0$. Let $\{\mathcal{F}_\varepsilon\}$ be the functionals introduced in (4.1), and let $\{u_\varepsilon\} \subseteq L^\infty(\mathbb{R}^n)$ be such that

$$\sup_{\varepsilon>0} \{\mathcal{F}_\varepsilon(u_\varepsilon, \mathbb{R}^n) + \|u_\varepsilon\|_\infty\} < +\infty. \tag{5.1}$$

Then there exist $\{\varepsilon_k\} \to 0^+$ and $u \in GSBV(\mathbb{R}^n)$ such that

$$\{u_{\varepsilon_k}\} \longrightarrow u \qquad \text{in } L^1_{loc}(\mathbb{R}^n).$$

Remark 5.2 If for some $p > 1$ the inequality in (Cpt1) is replaced by $\varphi_\varepsilon(r) \geq H_M r^p - K_M$, then the compactness result holds true in $L^p_{loc}(\mathbb{R}^n)$.

5.1 Proofs

In order to prove Theorem 5.1, let us introduce some notations. Let us assume that $\eta \in C^1(\mathbb{R}^n)$ is a non-zero non-negative function with compact support, and let $R > 0$ be such that

$$|\xi| \geq R \implies \eta(\xi) = 0.$$

Moreover we set

$$\omega_0 := \|\eta\|_{L^1(\mathbb{R}^n)}, \qquad \omega_1 := \|\nabla\eta\|_{L^1(\mathbb{R}^n)},$$

and we denote by $C^\delta u$ the convolutions

$$C^\delta u(x) \ := \ \frac{1}{\omega_0} \int_{\mathbb{R}^n} u(x + \delta\xi)\, \eta(\xi)\, d\xi, \tag{5.2}$$

defined for every $u \in L^\infty(\mathbb{R}^n)$, and every $\delta > 0$.

In a standard way it is possible to show that $C^\delta u \in C^1(\mathbb{R}^n)$ and moreover

$$\left\|C^\delta u\right\|_\infty \ \leq \ \|u\|_\infty, \qquad \left\|\nabla C^\delta u\right\|_\infty \ \leq \ \frac{\omega_1}{\delta}\, \|u\|_\infty. \tag{5.3}$$

We now need two technical lemmata.

Lemma 5.3 *Let $\{\varphi_\varepsilon\}$ be a family of Borel functions satisfying (Cpt1).*

Then for all $u \in L^\infty(\mathbb{R}^n)$, $\delta > 0$, $A \subset\subset \mathbb{R}^n$, we have that

$$\left\|C^\delta u - u\right\|_{L^1(A)} \ \leq \ \frac{R}{H_M\omega_0}\, (\omega_0 K_M |A| + \mathcal{F}_\delta(u, \mathbb{R}^n))\, \delta$$

for every $M \geq 2\|u\|_\infty$.

PROOF. Applying (Cpt1) with

$$\varepsilon = \delta|\xi|, \qquad r = \frac{|u(x+\delta\xi) - u(x)|}{\delta|\xi|},$$

we have that

$$\begin{aligned} |u(x+\delta\xi) - u(x)| &= \delta|\xi| \frac{|u(x+\delta\xi) - u(x)|}{\delta|\xi|} \\ &\leq \frac{\delta|\xi|}{H_M} \left\{ K_M + \varphi_{\delta|\xi|} \left(\frac{|u(x+\delta\xi) - u(x)|}{\delta|\xi|} \right) \right\} \end{aligned}$$

for every $M \geq 2\|u\|_\infty$. Therefore

$$\begin{aligned} \left\| C^\delta u - u \right\|_{L^1(A)} &\leq \frac{1}{\omega_0} \int_{A\times\mathbb{R}^n} |u(x+\delta\xi) - u(x)|\, \eta(\xi)\, dx\, d\xi \\ &\leq \frac{\delta R}{H_M \omega_0} \int_{\mathbb{R}^n} \int_A \left\{ K_M + \varphi_{\delta|\xi|} \left(\frac{|u(x+\delta\xi) - u(x)|}{\delta|\xi|} \right) \right\} dx\, \eta(\xi)\, d\xi \\ &\leq \frac{\delta R}{H_M \omega_0} \int_{\mathbb{R}^n} \left(K_M|A| + F_{\delta,\xi}(u,\mathbb{R}^n) \right) \eta(\xi)\, d\xi \\ &= \frac{R}{H_M \omega_0} \left(\omega_0 K_M |A| + \mathcal{F}_\delta(u,\mathbb{R}^n) \right) \delta. \quad \Box \end{aligned}$$

Lemma 5.4 *Let $\{\varphi_\varepsilon\}$ be a family of Borel functions satisfying (Cpt2). Then for all $\delta > 0$, $k \in \mathbb{N}\backslash\{0\}$, $u \in L^\infty(\mathbb{R}^n)$ we have that*

$$\mathcal{F}_{k\delta}(u,\mathbb{R}^n) \leq \mathcal{F}_\delta(u,\mathbb{R}^n). \tag{5.4}$$

PROOF. Let us argue by induction. If $k = 1$, thesis is trivial. Let us assume that (5.4) holds true for some $k \geq 1$. Applying (Cpt2) with $\varepsilon = \delta|\xi|$, and

$$A = \frac{u(x+(k+1)\delta|\xi|) - u(x+k\delta|\xi|)}{\delta|\xi|}, \quad B = \frac{u(x+k\delta|\xi|) - u(x)}{\delta|\xi|},$$

it follows that

$$\begin{aligned} &\varphi_{(k+1)\delta|\xi|} \left(\frac{(u(x+(k+1)\delta|\xi|) - u(x))}{(k+1)\delta|\xi|} \right) \leq \\ &\qquad \leq \frac{1}{k+1} \varphi_{\delta|\xi|} \left(\frac{(u(x+(k+1)\delta|\xi|) - u(x+k\delta|\xi|))}{\delta|\xi|} \right) + \\ &\qquad\quad + \frac{k}{k+1} \varphi_{k\delta|\xi|} \left(\frac{(u(x+k\delta|\xi|) - u(x))}{k\delta|\xi|} \right). \end{aligned}$$

Multiplying by $\eta(\xi)$ and integrating in (x,ξ) over $\mathbb{R}^n\times\mathbb{R}^n$, by the inductive hypothesis we obtain that

$$\mathcal{F}_{(k+1)\delta}(u,\mathbb{R}^n) \;\le\; \frac{1}{k+1}\,\mathcal{F}_{\delta}(u,\mathbb{R}^n) + \frac{k}{k+1}\,\mathcal{F}_{k\delta}(u,\mathbb{R}^n) \;\le\; \mathcal{F}_{\delta}(u,\mathbb{R}^n),$$

and this completes the induction. $\square$

Proof of Theorem 5.1.

Up to replacing η by a smaller function, we can assume that η belongs to $C^1(\mathbb{R})$ and has compact support (this is the point where we use our assumptions on η). Now we argue as in the case of the Mumford-Shah functional. We show that $\{u_{\varepsilon_j}\}$ is relatively compact in $L^1(A)$ for every sequence $\{\varepsilon_j\}\to 0^+$ and every $A\subset\subset\mathbb{R}^n$. To this end we set for every $\sigma>0$

$$K_\sigma := \left\{C^{\varepsilon_j[\frac{\sigma}{\varepsilon_j}]}u_{\varepsilon_j}\,:\,\varepsilon_j\le\frac{\sigma}{2}\right\}\cup\bigcup_{\varepsilon_j>\frac{\sigma}{2}}\{u_{\varepsilon_j}\},$$

and we show that

(i) K_σ is relatively compact in $L^1(A)$;

(ii) for all $j\in\mathbb{N}$, there exists $v_j\in K_\sigma$ such that $\|u_{\varepsilon_j}-v_j\|_{L^1(A)}\le N\,\sigma$, where N does not depend on j and σ.

This proves that the sequence $\{u_{\varepsilon_j}\}$ is totally bounded, hence relatively compact, in $L^1(A)$.

Let us show that K_σ satisfies (i). Since there is only a finite number of $\varepsilon_j>\sigma/2$, it suffices to show that

$$\tilde{K}_\sigma = \left\{C^{\varepsilon_j[\frac{\sigma}{\varepsilon_j}]}u_{\varepsilon_j}\,:\,\varepsilon_j\le\frac{\sigma}{2}\right\}$$

is relatively compact in $L^1(A)$. To this end, let us remark that $\tilde{K}_\sigma\subseteq C^1(A)$, and since $\varepsilon_j[\frac{\sigma}{\varepsilon_j}]\ge\frac{\sigma}{2}$ by (5.3) we have that

$$\left\|C^{\varepsilon_j[\frac{\sigma}{\varepsilon_j}]}u_{\varepsilon_j}\right\|_\infty \;\le\; \|u_{\varepsilon_j}\|_\infty,\qquad \left\|\nabla C^{\varepsilon_j[\frac{\sigma}{\varepsilon_j}]}u_{\varepsilon_j}\right\|_\infty \;\le\; \frac{2\omega_1}{\sigma}\,\|u_{\varepsilon_j}\|_\infty.$$

By Ascoli's Theorem, $\tilde{K}_\sigma$ is relatively compact in $C^0(A)$, hence in $L^1(A)$.

Let us show that K_σ satisfies (ii) with

$$N := \frac{R}{H_M \omega_0}\left(\omega_0 K_M |A| + \sup_{\varepsilon>0} \mathcal{F}_\varepsilon(u_\varepsilon, \mathbb{R}^n)\right), \qquad M := 2\sup_{\varepsilon>0} \|u_\varepsilon\|_\infty .$$

If $\varepsilon_j > \sigma/2$ we can simply take $v_j = u_{\varepsilon_j}$. If $\varepsilon_j \le \sigma/2$ we can take $v_j = C^{\varepsilon_j[\frac{\sigma}{\varepsilon_j}]} u_{\varepsilon_j}$. Indeed, by Lemma 5.3 and Lemma 5.4, we have that

$$\begin{aligned}
\left\| C^{\varepsilon_j[\frac{\sigma}{\varepsilon_j}]} u_{\varepsilon_j} - u_{\varepsilon_j} \right\|_{L^1(A)} &\le \frac{R}{H_M\omega_0}\left(\omega_0 K_M|A| + \mathcal{F}_{\varepsilon_j[\frac{\sigma}{\varepsilon_j}]}(u_{\varepsilon_j}, \mathbb{R}^n))\right)\sigma \\
&\le \frac{R}{H_M\omega_0}\left(\omega_0 K_M|A| + \mathcal{F}_{\varepsilon_j}(u_{\varepsilon_j}, \mathbb{R}^n))\right)\sigma \;\le\; N\sigma.
\end{aligned}$$

By (5.1) and the liminf inequality in the definition of Γ^--convergence, any limit point of $\{u_\varepsilon\}$ satisfies $\mathcal{F}_{\varphi,\psi}(u, \mathbb{R}^n) < +\infty$, hence belongs to $GSBV(\mathbb{R}^n)$. □

6 Approximation of Free Discontinuity Problems

In this section we prove that large classes of functionals like (2.2) can be approximated by non-local functionals of the form (4.1). To this end, we need the following definition.

Definition 6.1 We say that an increasing convex function $\varphi : [0, +\infty[\to [0, +\infty]$ is *sectionable* in $\mathbb{R}^n$ if there exists a convex function $\overline{\varphi} : [0, +\infty[\to [0, +\infty]$ such that

$$\varphi(|\alpha|) = \frac{1}{\mathcal{H}^{n-1}(S^{n-1})} \int_{S^{n-1}} \overline{\varphi}\left(|\langle \alpha, v\rangle|\right) d\mathcal{H}^{n-1}(v) \qquad \forall\, \alpha \in \mathbb{R}^n. \tag{6.1}$$

Remark 6.2 The following properties of sectionable functions are an immediate consequence of the above definition.

- Every convex function $\varphi : [0, +\infty[\to [0, +\infty]$ is sectionable in $\mathbb{R}$, and $\overline{\varphi} = \varphi$.

- For every real number $p \geq 1$, the function $\varphi(r) = r^p$ is sectionable in $\mathbb{R}^n$ for every n. Indeed, (6.1) is satisfied with $\overline{\varphi}(r) = (c_{p,n})^{-1} r^p$, where $c_{p,n}$ is the constant introduced in (4.12).
- The class of sectionable functions is additively closed. Moreover, if φ is sectionable, and $\lambda > 0$ is a constant, then $\lambda\varphi$ is sectionable.
- The class of sectionable functions is closed by monotone convergence in the following sense: if $\{\varphi_n\}$ is a sequence of sectionable functions, and $\varphi_{n+1}(r) \geq \varphi_n(r)$ for every $n \in \mathbb{N}$ and every $r \geq 0$, then $\sup_n \varphi_n$ is sectionable. In this way we can show, for example, that $\varphi(r) = e^{r^2}$ is sectionable.
- Every sectionable function φ is the supremum of an increasing sequence of sectionable *finite* functions (it is enough to approximate $\overline{\varphi}$ with an increasing sequence of finite convex functions).
- If φ is sectionable in $\mathbb{R}^n$ and satisfies (2.3), then also $\overline{\varphi}$ satisfies (2.3).
- It can be proved that $\varphi(r) := \max\{0, r-1\}$ is *not* sectionable in $\mathbb{R}^n$ for every $n > 1$.

The following is the main result of this paper.

Theorem 6.3 *Let φ and ψ be as in the lower semicontinuity Theorem 2.1, and let $\eta \in L^1(\mathbb{R}^n)$ be a non-negative radial function such that $\{\xi \in \mathbb{R}^n : \eta(\xi) > c\}$ has non-empty interior for some $c > 0$. Let us assume that φ is sectionable in $\mathbb{R}^n$.*

Then there exists a family $\{\varphi_\varepsilon\}$ such that, defining $\{\mathcal{F}_\varepsilon\}$ as in (4.1), we have that

(C1) $\mathcal{F}_\varepsilon(u, \mathbb{R}^n) \leq \mathcal{F}_{\varphi,\psi}(u, \mathbb{R}^n)$ *for all* $u \in L^1_{loc}(\mathbb{R}^n)$, *and all* $\varepsilon > 0$;

(C2) $\{\mathcal{F}_\varepsilon(u, \mathbb{R}^n)\}$ *converges to* $\mathcal{F}_{\varphi,\psi}(u, \mathbb{R}^n)$ *for all* $u \in L^1_{loc}(\mathbb{R}^n)$;

(C3) $\mathcal{F}_{\varphi,\psi}(u, \mathbb{R}^n)$ *is the* Γ^-*-limit of* $\{\mathcal{F}_\varepsilon(u, \mathbb{R}^n)\}$ *in* $L^1_{loc}(\mathbb{R}^n)$;

(C4) *if* $\{u_\varepsilon\} \subseteq L^\infty(\mathbb{R}^n)$ *and*

$$\sup_{\varepsilon > 0} \{\mathcal{F}_\varepsilon(u_\varepsilon, \mathbb{R}^n) + \|u_\varepsilon\|_\infty\} \; < \; +\infty,$$

then there exist $\{\varepsilon_k\} \to 0^+$ *and* $u \in GSBV(\mathbb{R}^n)$ *such that*

$$\{u_{\varepsilon_k}\} \longrightarrow u \qquad \text{in } L^1_{loc}(\mathbb{R}^n).$$

Remark 6.4 For simplicity's sake we developed our theory under the assumptions of Theorem 2.1, as stated in [3, 4]. However, it is well known that Theorem 2.1 holds true also when the assumption "ψ is concave" is relaxed to "ψ is sub-additive and lower semi-continuous" (see *e.g.* [8, Theorem 2.10]). In the same way, throughout all this paper (hence in Theorem 6.3 above), we can modify the concavity assumptions on ψ to sub-additivity and lower semi-continuity (but some proofs may become longer!).

Remark 6.5 The family $\{\varphi_\varepsilon\}$ given by Theorem 6.3 is clearly not unique. In our proof, φ_ε will be defined as the minimum of a family of functions. This construction is convenient from the theoretic point of view, but often it is difficult to give an explicit expression of this minimum. For this reason, in many applications it may be useful to find other families with a simpler analytic expression, and then prove the convergence case-by-case using Theorem 4.3 and Theorem 5.1 (cf. the examples in § 7).

6.1 Proofs

In this subsection we prove Theorem 6.3. To this end, we need two lemmata about real functions.

Lemma 6.6 *Let $f : [0,+\infty[\to [0,+\infty[$ be a non-decreasing convex function, and let $g : [0,+\infty] \to [0,+\infty[$ be a non-decreasing concave function such that $g(0) = 0$. Let us set*

$$\mu(r) := \min\{f(l) + g(r-l) : \; l \in [0,r]\}. \tag{6.2}$$

Then

(i) *μ is continuous and non-decreasing;*

(ii) *there exists $\overline{r} \geq 0$ such that μ is convex in $[0,\overline{r}]$ and concave in $[\overline{r},+\infty[$.*

Proof. Let us first remark that our assumptions imply the continuity of f in $[0,+\infty[$, and the continuity of g in $]0,+\infty[$ (but not the continuity of g in $[0,+\infty[\,!$). In any case, g is at least lower semicontinuous in $[0,+\infty[$, and therefore the "min" in (6.2) is attained. Moreover, by definition of μ we have that

$$\mu(r) \leq f(r), \qquad \forall\, r \geq 0. \tag{6.3}$$

We claim that μ satisfies (ii) with

$$\overline{r} := \sup\{r \geq 0 : \ \mu(r) = f(r)\}.$$

Step 1. We prove that μ is non-decreasing. To this end, let $s > r$, and let $l \in [0, s]$ be such that $\mu(s) = f(l) + g(s - l)$.

If $l \in [0, r]$, by the monotonicity of g we have that

$$\mu(s) \ = \ f(l) + g(s-l) \ \geq \ f(l) + g(r-l) \ \geq \ \mu(r).$$

If $l \in]r, s]$, by the monotonicity of f and (6.3) it follows that

$$\mu(s) \ = \ f(l) + g(s-l) \ \geq \ f(r) \ \geq \ \mu(r).$$

In any case, we have proved that $\mu(s) \geq \mu(r)$.

Step 2. We show that

$$\mu(r) = f(r), \qquad \forall\, r \in [0, \overline{r}], \tag{6.4}$$

and therefore μ is convex in $[0, \overline{r}]$.

Indeed, let us assume by contradiction that $\mu(r_*) < f(r_*)$ for some $r_* \in [0, \overline{r}[$. Then, there exists $l < r_*$ such that

$$f(l) + g(r_* - l) < f(r_*).$$

Now let us consider the function $\gamma : [l, +\infty[\to \mathbb{R}$ defined by

$$\gamma(t) := f(t) - g(t - l) - f(l).$$

Since γ is convex, $\gamma(l) = 0$, and $\gamma(r_*) > 0$, then necessarily $\gamma(t) > 0$ for every $t \geq r_*$. Therefore

$$\mu(r) \ \leq \ f(l) + g(r-l) \ < \ f(r) \qquad \forall\, r \geq r_*,$$

which contradicts the definition of $\overline{r}$. This proves that $\mu(r) = f(r)$ for every $r \in [0, \overline{r}[$. By the monotonicity of μ and (6.3) it follows that

$$f(r) \ = \ \mu(r) \ \leq \ \mu(\overline{r}) \ \leq f(\overline{r})$$

for every $r < \overline{r}$. Passing to the limit as $r \to \overline{r}^-$, the proof of (6.4) is complete.

Step 3. We prove that there exists $\bar{l} \in [0, \bar{r}]$ such that

$$f\left(\bar{l}\right) + g\left(r - \bar{l}\right) \;\leq\; f(r), \qquad \forall\, r \geq \bar{r}. \tag{6.5}$$

Indeed, let $\{r_n\} \to \bar{r}^+$ be any sequence, and for each $n \in \mathbb{N}$, let $l_n \in [0, r_n]$ be such that

$$\mu(r_n) \;=\; f(l_n) + g(r_n - l_n) \;<\; f(r_n),$$

where the inequality follows from the definition of $\bar{r}$.

Up to subsequences, we can assume that $\{l_n\} \to \bar{l} \in [0, \bar{r}]$. In order to prove that (6.5) holds true, let us fix $r > \bar{r}$, and let us consider the functions $\gamma_n : [l_n, +\infty[\to \mathbb{R}$ defined by

$$\gamma_n(t) \;:=\; f(t) - g(t - l_n) - f(l_n).$$

Since γ_n is a convex function such that $\gamma_n(l_n) = 0$ and $\gamma_n(r_n) > 0$, then necessarily $\gamma_n(t) > 0$ for every $t \geq r_n$. Since $r \geq r_n$ for n large enough, it follows that $\gamma_n(r) > 0$ for n large enough. Passing to the limit as $n \to +\infty$ we obtain that

$$f(r) - g(r - \bar{l}) - f(\bar{l}) \geq 0,$$

which is equivalent to (6.5).

Step 4. We prove that

$$\mu(r) = \min\{f(l) + g(r - l) :\; l \in [0, \bar{r}]\}, \qquad \forall\, r \geq \bar{r}. \tag{6.6}$$

Indeed, if $r \geq l \geq \bar{r}$, then, using (6.5) with $r = l$, and the subadditivity of g, it follows that

$$f(l) + g(r - l) \;\geq\; f(\bar{l}) + g(l - \bar{l}) + g(r - l) \;\geq\; f(\bar{l}) + g(r - \bar{l}).$$

This proves that for $r \geq \bar{r}$, in the minimum problem (6.2) we can consider only the values $l \in [0, \bar{r}]$.

Step 5. By (6.6) we have that for $r \geq \bar{r}$, the function μ is the minimum of a *fixed* family of concave functions. This proves that μ is concave in $[\bar{r}, +\infty[$.

Step 6. In order to complete the proof of the lemma, it remains to show that μ is continuous.

By (6.4) the restriction of μ to $[0, \bar{r}]$ is continuous. Moreover, μ is continuous on $]\bar{r}, +\infty[$ since it is concave in this region. Therefore it remains to prove that

$$\lim_{r \to \bar{r}^+} \mu(r) = \mu(\bar{r}). \tag{6.7}$$

By the monotonicity of μ, and (6.5), it follows that

$$f(\overline{r}) \;=\; \mu(\overline{r}) \;\leq\; \mu(r) \;\leq\; f(\overline{l}) + g(r - \overline{l}) \;\leq\; f(r) \qquad\qquad \forall\, r \geq \overline{r}.$$

Passing to the limit as $r \to \overline{r}^+$, (6.7) is proved. $\square$

Lemma 6.7 *Let $\varphi : [0, +\infty[\to [0, +\infty[$ be a non-decreasing convex function not identically zero, and let $\psi : [0, +\infty] \to [0, +\infty[$ be a non-decreasing concave function such that $\psi(0) = 0$. Let us set*

$$\varphi_\varepsilon(r) \;:=\; \min\left\{\varphi(l) + \frac{1}{\varepsilon}\,\psi(\varepsilon(r - l)) :\; l \in [0, r]\right\}. \tag{6.8}$$

for every $\varepsilon > 0$. Then

(i) the family $\{\varphi_\varepsilon\}$ satisfies (li1), (li2), (Est), (Cpt1) and (Cpt2).

If moreover φ and ψ satisfy (2.3) and (2.4), then

(ii) $\{\varphi_\varepsilon(r)\} \to \varphi(r)$ uniformly on compact subsets of $[0, +\infty[$;

(iii) $\{\varepsilon\varphi_\varepsilon(r/\varepsilon)\} \to \psi(r)$ uniformly on compact subsets of $]0, +\infty[$.

In particular

$$\varphi(r) = \Gamma^{-}\text{-}\liminf_{\varepsilon \to 0^+}\, \varphi_\varepsilon(r), \qquad \psi(r) = \Gamma^{-}\text{-}\liminf_{\varepsilon \to 0^+}\, \varepsilon\varphi_\varepsilon\left(\frac{r}{\varepsilon}\right).$$

Proof.
Proof of (i). Properties (li1) and (li2) follow from Lemma 6.6 applied with $f(r) = \varphi(r)$ and $g(r) = \psi(\varepsilon r)/\varepsilon$. Property (Est) is a trivial consequence of the definition (6.8).

Since φ is convex and non-zero, then there exists $c > 0$ and $d \geq 0$ such that

$$\varphi(r) \;\geq\; cr - d \qquad \forall\, r \geq 0.$$

Moreover, since ψ is concave, then for every $M > 0$ we have that

$$\psi(r) \;\geq\; \frac{\psi(M)}{M}\, r \qquad \forall\, r \in [0, M].$$

We claim that φ_ε satisfies (Cpt1) with

$$H_M := \min\left\{\frac{\psi(M)}{M}, c\right\}, \qquad K_M = d.$$

Indeed, for every $0 \le l \le r \le M/\varepsilon$ we have that $\varepsilon(r-l) \le M$, hence

$$\varphi(l) + \frac{1}{\varepsilon}\,\psi(\varepsilon(r-l)) \;\ge\; H_M l - K_M + \frac{1}{\varepsilon}\,H_M\varepsilon(r-l) \;=\; H_M\,r - K_M.$$

This is equivalent to (Cpt1).

Now let us prove that $\{\varphi_\varepsilon\}$ satisfies (Cpt2). Let $l_A \in [0, A]$ and $l_B \in [0, B/k]$ be such that

$$\varphi_\delta(A) \;=\; \varphi(l_A) + \frac{1}{\delta}\,\psi(\delta(A - l_A)),$$

$$\varphi_{k\delta}\left(\frac{B}{k}\right) \;=\; \varphi(l_B) + \frac{1}{k\delta}\,\psi\left(k\delta\left(\frac{B}{k} - l_B\right)\right).$$

Then

$$l_C \;:=\; \frac{l_A + k l_B}{k+1} \in \left[0, \frac{A+B}{k+1}\right],$$

hence, by the convexity of φ and the subadditivity of ψ, it follows that

$$\begin{aligned}
\varphi_{(k+1)\delta}\left(\frac{A+B}{k+1}\right) \;&\le\; \varphi\left(l_C\right) + \frac{1}{(k+1)\delta}\,\psi\left((k+1)\delta\left(\frac{A+B}{k+1} - l_C\right)\right)\\
&=\; \varphi\left(\frac{l_A}{k+1} + \frac{k}{k+1} l_B\right) + \frac{1}{(k+1)\delta}\,\psi\left(\delta(A - l_A) + k\delta\left(\frac{B}{k} - l_B\right)\right)\\
&\le\; \frac{1}{k+1}\,\varphi(l_A) + \frac{k}{k+1}\,\varphi(l_B)\\
&\quad + \frac{1}{(k+1)\delta}\,\psi(\delta(A - l_A)) + \frac{k}{(k+1)}\,\frac{1}{k\delta}\,\psi\left(k\delta\left(\frac{B}{k} - l_B\right)\right)\\
&=\; \frac{1}{k+1}\,\varphi_\delta(A) + \frac{k}{k+1}\,\varphi_{k\delta}\left(\frac{B}{k}\right).
\end{aligned}$$

Proof of (ii). Let $r \ge 0$. Setting $l = r$ in (6.8) we have that

$$\varphi_\varepsilon(r) \le \varphi(r). \tag{6.9}$$

Moreover, let $l_\varepsilon \in [0, r]$ be such that

$$\varphi_\varepsilon(r) = \varphi(l_\varepsilon) + \frac{1}{\varepsilon}\,\psi(\varepsilon(r - l_\varepsilon)). \tag{6.10}$$

We claim that $\{l_\varepsilon\} \to r$. Indeed, let us assume by contradiction that there exists a sequence $\{\varepsilon_n\} \to 0^+$ such that $\{r - l_{\varepsilon_n}\} \to \alpha > 0$. Then, since $\{\varepsilon_n(r - l_{\varepsilon_n})\} \to 0$, by (6.10) and (2.4) it follows that

$$\begin{aligned} \lim_{n\to+\infty} \varphi_{\varepsilon_n}(r) &= \lim_{n\to+\infty} \varphi(l_{\varepsilon_n}) + \frac{1}{\varepsilon_n}\,\psi(\varepsilon_n(r - l_{\varepsilon_n})) \\ &\geq \lim_{n\to+\infty} (r - l_{\varepsilon_n})\,\frac{\psi(\varepsilon_n(r - l_{\varepsilon_n}))}{\varepsilon_n(r - l_{\varepsilon_n})} \;=\; +\infty, \end{aligned}$$

which is impossible because of (6.9). Since $\{l_\varepsilon\} \to r$, then by (6.10) it follows that

$$\liminf_{\varepsilon\to 0^+} \varphi_\varepsilon(r) \;\geq\; \liminf_{\varepsilon\to 0^+} \varphi(l_\varepsilon) \;=\; \varphi(r),$$

which, together with (6.9), proves that $\{\varphi_\varepsilon(r)\} \to \varphi(r)$ for all $r \geq 0$. Since φ_ε and φ are continuous increasing functions, uniform convergence on compact subsets follows from pointwise convergence.

Proof of (iii). Let $r > 0$. Setting $l = 0$ in (6.8) we have that

$$\varepsilon\varphi_\varepsilon\left(\frac{r}{\varepsilon}\right) \leq \psi(r) + \varepsilon\varphi(0). \tag{6.11}$$

Moreover, let $l_\varepsilon \in [0, r/\varepsilon]$ be such that

$$\varphi_\varepsilon\left(\frac{r}{\varepsilon}\right) = \varphi(l_\varepsilon) + \frac{1}{\varepsilon}\,\psi(r - \varepsilon l_\varepsilon). \tag{6.12}$$

We claim that $\{\varepsilon l_\varepsilon\} \to 0$. Indeed, let us assume by contradiction that there exists a sequence $\{\varepsilon_n\} \to 0^+$ such that $\{\varepsilon_n l_{\varepsilon_n}\} \to \alpha \in]0, r]$. Then $\{l_{\varepsilon_n}\} \to +\infty$ hence, by (6.12) and (2.3),

$$\begin{aligned} \lim_{n\to+\infty} \varepsilon_n\varphi_{\varepsilon_n}\left(\frac{r}{\varepsilon_n}\right) &= \lim_{n\to+\infty} \varepsilon_n\varphi(l_{\varepsilon_n}) + \psi(r - \varepsilon_n l_{\varepsilon_n}) \\ &\geq \lim_{n\to+\infty} \varepsilon_n l_{\varepsilon_n}\,\frac{\varphi(l_{\varepsilon_n})}{l_{\varepsilon_n}} = +\infty, \end{aligned}$$

which is impossible because of (6.11). Since $\{\varepsilon l_\varepsilon\} \to 0$, then by (6.12) it follows that

$$\liminf_{\varepsilon\to 0^+} \varepsilon\varphi_\varepsilon\left(\frac{r}{\varepsilon}\right) \;\geq\; \liminf_{\varepsilon\to 0^+} \psi(r-\varepsilon l_\varepsilon) \;=\; \psi(r),$$

which, together with (6.11), proves that $\{\varepsilon\varphi_\varepsilon(r/\varepsilon)\} \to \psi(r)$ for every $r > 0$. Since $\varepsilon\varphi_\varepsilon(r/\varepsilon)$ and $\psi(r)$ are continuous increasing functions, uniform convergence on compact subsets follows from pointwise convergence. $\square$

Proof of Theorem 6.3.

Let $\eta(\xi) = J(|\xi|)$ for some $J : [0,+\infty[\to [0,+\infty[$. Let us assume that φ and ψ are finite. In this case, we extend ψ to $[0,+\infty]$ by setting $\psi(0) = 0$, and then, for every $\varepsilon > 0$, we define

$$\varphi_\varepsilon(r) := \frac{1}{\omega}\min\left\{\overline{\varphi}(l) + \frac{1}{\varepsilon}\,\psi(\varepsilon(r-l)) :\; l \in [0,r]\right\}, \qquad \forall\, r \geq 0,$$

where $\overline{\varphi}$ is given by (6.1), and $\omega := \|\eta\|_{L^1(\mathbb{R}^n)}$.

By Lemma 6.7 it follows that $\{\varphi_\varepsilon\}$ satisfies (li1), (li2) and (Est) with

$$\varphi_\star(r) = \varphi^\star(r) = \frac{\overline{\varphi}(r)}{\omega}, \qquad \psi_\star(r) = \psi^\star(r) = \frac{\psi(r)}{\omega}.$$

Since using spherical coordinates in (4.11) we have that

$$\begin{aligned}
\left(S(\overline{\varphi}/\omega)\right)(\alpha) \;&=\; \frac{1}{\omega}\int_{\mathbb{R}^n} \frac{1}{\omega}\,\overline{\varphi}\left(|\langle \alpha, \xi/|\xi|\rangle|\right)\eta(\xi)\,d\xi \\
&=\; \frac{1}{\omega^2}\int_0^{+\infty} \rho^{n-1}\,J(\rho)\,d\rho \int_{S^{n-1}} \overline{\varphi}(|\langle \alpha, v\rangle|)\,d\mathcal{H}^{n-1}(v) \\
&=\; \frac{1}{\omega^2}\int_0^{+\infty} \rho^{n-1}\,J(\rho)\,\mathcal{H}^{n-1}(S^{n-1})\,\varphi(|\alpha|)\,d\rho \;=\; \frac{1}{\omega}\,\varphi(|\alpha|),
\end{aligned}$$

then statements (C1), (C2), and (C3) follow from Theorem 4.3.

Moreover, by Lemma 6.7 we have that $\{\varphi_\varepsilon\}$ satisfies also (Cpt1) and (Cpt2), hence statement (C4) follows from Theorem 5.1.

If φ and ψ are not finite, then we first approximate $\mathcal{F}_{\varphi,\psi}$ from below by functionals $\{\mathcal{F}_{\varphi_n,\psi_n}\}$, where φ_n and ψ_n are finite functions satisfying the assumptions of this theorem. Arguing as before, we approximate the functionals $\mathcal{F}_{\varphi_n,\psi_n}$, and then we conclude the proof by a diagonal argument. $\square$

7 Examples

In this section we give some applications of the results proved in the previous sections. We apply Theorem 4.3 and Theorem 5.1 in order to prove the convergence results (C1) through (C4) of § 1 for some special choices of $\{\varphi_\varepsilon\}$.

From now on, we assume that $J : [0,+\infty[\to [0,+\infty[$ is a continuous function not identically zero such that

$$j_\alpha := \int_0^{+\infty} \rho^{\alpha-1} J(\rho)\, d\rho$$

is finite for each real number $\alpha \geq 1$. We also consider the constants $c_{p,n}$ defined in (4.12). In particular: $c_{0,n} = \mathcal{H}^{n-1}(S^{n-1})$, $J(|\xi|) \in L^1(\mathbb{R}^n)$, and $\|\eta\|_{L^1(\mathbb{R}^n)} = c_{0,n}\, j_n$.

Example 7.1 Let us consider the functionals

$$\mathcal{F}_\varepsilon(u) \;:=\; \frac{1}{\varepsilon^p} \int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x+\varepsilon\xi) - u(x)|^p \, J(|\xi|)\, d\xi\, dx,$$

with $p > 1$. Then $\{\mathcal{F}_\varepsilon\}$ satisfies (C1) through (C4) of § 1 with

$$\mathcal{F}(u) \;:=\; \begin{cases} \lambda \displaystyle\int_{\mathbb{R}^n} |\nabla u(x)|^p dx & \text{if } u \in W^{1,p}_{loc}(\mathbb{R}^n), \\ +\infty & \text{if } u \in L^1_{loc}(\mathbb{R}^n) \setminus W^{1,p}_{loc}(\mathbb{R}^n), \end{cases}$$

where $\lambda = c_{p,n}\, j_{p+n}$.

Indeed the family $\{\mathcal{F}_\varepsilon\}$ is a particular instance of (4.1) with

$$\varphi_\varepsilon(r) \;:=\; |r|^p, \qquad \eta(\xi) \;:=\; |\xi|^p J(|\xi|).$$

Since φ_ε satisfies (li1), (li2), (Est), (Cpt1), (Cpt2) with

$$\varphi_\star(r) = \varphi^\star(r) = |r|^p, \qquad \psi_\star(r) = \psi^\star(r) = +\infty,$$

then the results follow from Theorem 4.3 and Theorem 5.1.

Example 7.2 Let us consider the functionals

$$\mathcal{F}_\varepsilon(u) \;:=\; \frac{1}{\varepsilon}\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x+\varepsilon\xi)-u(x)|^{1/p}\, J(|\xi|)\, d\xi\, dx,$$

with $p>1$. Then $\{\mathcal{F}_\varepsilon\}$ satisfies (C1) through (C4) of § 1 with

$$\mathcal{F}(u) \;:=\; \begin{cases} \lambda \displaystyle\int_{S_u} |u^+-u^-|^{1/p}\, d\mathcal{H}^{n-1} & \text{if } u\in GSBV(\mathbb{R}^n)\cap L^1_{loc}(\mathbb{R}^n) \text{ and} \\ & \nabla u(x)=0 \text{ for a.e. } x\in\mathbb{R}^n, \\ +\infty & \text{if } u\in L^1_{loc}(\mathbb{R}^n)\setminus GSBV(\mathbb{R}^n), \end{cases}$$

where $\lambda = c_{0,n}\, j_{n+1/p}$.

Indeed the family $\{\mathcal{F}_\varepsilon\}$ is a particular instance of (4.1) with

$$\varphi_\varepsilon(r) \;:=\; \frac{1}{\varepsilon}\,|\varepsilon r|^{1/p}, \qquad\qquad \eta(\xi) \;:=\; |\xi|^{1/p} J(|\xi|).$$

Since φ_ε satisfies (li1), (li2), (Est), (Cpt1), (Cpt2) with

$$\varphi_\star(r)=\varphi^\star(r)=+\infty, \qquad\qquad \psi_\star(r)=\psi^\star(r)=|r|^{1/p},$$

then the results follow from Theorem 4.3 and Theorem 5.1.

Example 7.3 Let us consider the functionals

$$\mathcal{F}_\varepsilon(u) \;:=\; \frac{1}{\varepsilon}\int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x+\varepsilon\xi)-u(x)|\, J(|\xi|)\, d\xi\, dx.$$

This is the limit case $p=1$ both of Example 7.1 and of Example 7.2. In this case (C1) through (C4) of § 1 are satisfied with

$$\mathcal{F}(u) \;:=\; \lambda\, |Du|\,(\mathbb{R}^n),$$

where $\lambda = c_{1,n}\, j_{n+1}$, and $|Du|\,(\mathbb{R}^n)$ is defined as in (2.1).

Indeed the family $\{\mathcal{F}_\varepsilon\}$ is a particular instance of (4.1) with

$$\varphi_\varepsilon(r) \;:=\; |r|, \qquad\qquad \eta(\xi) \;:=\; |\xi| J(|\xi|).$$

It is easy to verify that φ_ε satisfies (li1), (li2), (Est), (Cpt1), (Cpt2) with

$$\varphi_\star(r)=\varphi^\star(r)=r, \qquad\qquad \psi_\star(r)=\psi^\star(r)=r.$$

Therefore the compactness property (C4) follows from Theorem 5.1. In order to prove (C1), (C2), (C3) we cannot apply directly Theorem 4.3, since φ and ψ do not satisfy (2.3) and (2.4). However we can apply Remark 4.4. Since for $\varphi(r) = \psi(r) = r$ it is well known that $\overline{\mathcal{F}_{\varphi,\psi}}(v, \mathbb{R}) = |Dv|(\mathbb{R})$ for every $v \in L^1_{loc}(\mathbb{R})$, by (4.13) it follows that (C1), (C2), (C3) are satisfied with

$$\tilde{\mathcal{F}}(u) \;=\; \int_{\mathbb{R}^n} \left(\int_{\langle\xi\rangle^\perp} |Du_{\xi,y}|\,(\mathbb{R})\, dy \right) \eta(\xi)\, d\xi \;=\; \lambda\, |Du|\,(\mathbb{R}^n),$$

where the last equality follows from a standard integral geometric computation.

Example 7.4 Let us consider the functionals

$$\mathcal{F}_\varepsilon(u) \;:=\; \frac{1}{\varepsilon} \int_{\mathbb{R}^n\times\mathbb{R}^n} \arctan\left(\frac{|u(x+\varepsilon\xi) - u(x)|^2}{\varepsilon|\xi|} \right) J(|\xi|)\, d\xi\, dx.$$

Then $\{\mathcal{F}_\varepsilon\}$ satisfies (C1) through (C4) of § 1 with

$$\mathcal{F}(u) \;:=\; \begin{cases} \lambda \displaystyle\int_{\mathbb{R}^n} |\nabla u|^2 dx + \mu\, \mathcal{H}^{n-1}(S_u) & \text{if } u \in GSBV(\mathbb{R}^n) \cap L^1_{loc}(\mathbb{R}^n), \\ +\infty & \text{if } u \in L^1_{loc}(\mathbb{R}^n) \setminus GSBV(\mathbb{R}^n), \end{cases}$$

where $\lambda = c_{2,n}\, j_{n+1}$, and $\mu = \frac{\pi}{2}\, c_{0,n}\, j_{n+1}$. This family is very similar to the family $\{\mathcal{DG}_\varepsilon\}$ studied in [17]. In this case we have that

$$\varphi_\varepsilon(r) \;:=\; \frac{1}{\varepsilon} \arctan(\varepsilon r^2), \qquad\qquad \eta(\xi) \;:=\; |\xi| J(|\xi|).$$

Since φ_ε satisfies (li1), (li2), (Est), (Cpt1), (Cpt2) with

$$\varphi_\star(r) = \varphi^\star(r) = |r|^2, \qquad\qquad \psi_\star(r) = \psi^\star(r) = \frac{\pi}{2},$$

then the results follow from Theorem 4.3 and Theorem 5.1.

Example 7.5 Let us consider the functionals

$$\mathcal{F}_\varepsilon(u) \;:=\; \frac{1}{\varepsilon} \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|u(x+\varepsilon\xi) - u(x)|^2}{|u(x+\varepsilon\xi) - u(x)|^{3/2} + \varepsilon|\xi|}\, J(|\xi|)\, d\xi\, dx.$$

Then $\{\mathcal{F}_\varepsilon\}$ satisfies (C1) through (C4) of § 1 with

$$\mathcal{F}(u) := \begin{cases} \lambda \displaystyle\int_{\mathbb{R}^n} |\nabla u(x)|^2 \, dx + \mu \int_{S_u} \sqrt{|u^+ - u^-|} \, d\mathcal{H}^{n-1} & \text{if } u \in GSBV(\mathbb{R}^n), \\ +\infty & \text{otherwise,} \end{cases}$$

where $\lambda = c_{2,n} \, j_{n+1}$, and $\mu = c_{0,n} \, j_{n+1}$.

Indeed the family $\{\mathcal{F}_\varepsilon\}$ is a particular instance of (4.1) with

$$\varphi_\varepsilon(r) := \frac{|r|^2}{\sqrt{\varepsilon}|r|^{3/2} + 1}, \qquad \eta(\xi) := |\xi| J(|\xi|).$$

Since it can be proved (exercise for the interested reader!) that φ_ε satisfies (li1), (li2), (Est) with

$$\varphi_\star(r) = \varphi^\star(r) = |r|^2, \qquad \psi_\star(r) = \psi^\star(r) = \sqrt{r},$$

then (C1), (C2), (C3) follow from Theorem 4.3.

If we want to prove (C4) applying directly Theorem 5.1, we are forced to show that φ_ε satisfies (Cpt2), but in this case this leads to a huge inequality which seems difficult to prove (or disprove). However, we can pursue a different path. We first remark that

$$\varphi_\varepsilon(r) \ \geq \ \tilde{\varphi}_\varepsilon(r) \ := \ \frac{1}{9} \min \left\{ l^2 + \frac{\sqrt{r-l}}{\sqrt{\varepsilon}} \, : \, l \in [0,r] \right\} \tag{7.1}$$

(use $l = r$ if $r \leq 4\varepsilon^{-1/3}$, and $l = 0$ otherwise), and that $\{\tilde{\varphi}_\varepsilon\}$ satisfies (Cpt1) and (Cpt2) (use Lemma 6.7). Then we consider the functional $\tilde{\mathcal{F}}_\varepsilon$ defined as in (3.1) with $\tilde{\varphi}_\varepsilon$ instead of φ_ε. Since by (7.1) we have that

$$\sup_{\varepsilon>0} \{\mathcal{F}_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty\} < +\infty \implies \sup_{\varepsilon>0} \{\tilde{\mathcal{F}}_\varepsilon(u_\varepsilon) + \|u_\varepsilon\|_\infty\} < +\infty,$$

then property (C4) for $\{\mathcal{F}_\varepsilon\}$ follows applying Theorem 5.1 to the family $\{\tilde{\mathcal{F}}_\varepsilon\}$.

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